Caps on Hermitian varieties and maximal curves

Article (Unspecified)

Caps on Hermitian varieties and maximal curves

J.W.P. Hirschfeld and G. Korchmáros

Dedicated to Adriano Barlotti on the occasion of his 80-th birthday

Abstract

A lower bound for the size of a complete cap of the polar space $H(n, q^2)$ associated to the non-degenerate Hermitian variety $\mathcal{U}_n$ is given; this turns out to be sharp for even q when $n = 3$. Also, a family of caps of $H(n, q^2)$ is constructed from $\mathbf{F}_{q^2}$-maximal curves. Such caps are complete for q even, but not necessarily for q odd.

1 Introduction

Let $\mathcal{U}_n$ be the non-degenerate Hermitian variety of the n-dimensional projective space $\mathrm{PG}(n, q^2)$ coordinatised by the finite field $\mathbf{F}_{q^2}$ of square order q^2. An *ovoid* of the polar space $H(n, q^2)$ arising from the non-degenerate Hermitian variety $\mathcal{U}_n$ with $n \geq 3$ is defined to be a point set in $\mathcal{U}_n$ having exactly one common point with every generator of $\mathcal{U}_n$. For n even, $\mathcal{U}_n$ has no ovoid; see [24]. For n odd, the existence problem for ovoids of $\mathcal{U}_n$ has been solved so far only in the smallest case $n = 3$; see [20].

A natural generalization of an ovoid is a *cap* (also called a *partial ovoid*). A cap of $\mathcal{U}_n$ is a point set in $\mathcal{U}_n$ which has at most one common point with every generator of $\mathcal{U}_n$. Equivalently, a cap is a point set consisting of pairwise non–conjugate points of $\mathcal{U}_n$. A cap is called *complete* if it is not contained in a larger cap of $\mathcal{U}_n$.

The size of a cap is at most $q^n + 1$ for odd n and q^n for even n; equality holds if and only if the cap is an ovoid. The following upper bound for the size k of a cap different from an ovoid is due to Moorhouse [19]:

$$k \leq \left[\binom{p+n-1}{n}^2 - \binom{p+n-2}{n}^2 \right]^h + 1, \qquad q = p^h. \tag{1.1}$$

A lower bound for k is given in Section 2 by proving that $k \geq q^2 + 1$.

In this paper a family of caps of $\mathcal{U}_n$ that are not ovoids is constructed, and it is shown that they are complete provided that $n = 3$ and q is even. The construction relies on an interesting property of $\mathbf{F}_{q^2}$–maximal curves of $\mathrm{PG}(n, q^2)$ that is stated in §3: the $\mathbf{F}_{q^2}$–rational points of an $\mathbf{F}_{q^2}$-maximal curve naturally embedded in a Hermitian variety $\mathcal{U}_n$ are pairwise non-conjugate under the associated unitary polarity. Hence the set $\mathcal{X}(\mathbf{F}_{q^2})$ of all $\mathbf{F}_{q^2}$–rational points of an $\mathbf{F}_{q^2}$–maximal curve is a cap of $\mathcal{U}_n$. The main result is that $\mathcal{X}(\mathbf{F}_{q^2})$ is a complete cap for $n = 3$ and q even.

For $n = 3$ and q odd, there exist $\mathbf{F}_{q^2}$-maximal curves such that $\mathcal{X}(\mathbf{F}_{q^2})$ is a cap of size $\frac{1}{2}(q^3 - q)$ contained in an ovoid of $\mathcal{U}_3$; see Example 4.8.

2 A lower bound for the size of a complete cap of $\mathcal{U}_n$

A (non–degenerate) Hermitian variety $\mathcal{U}_n$ is defined as the set of all self–conjugate points of a non–degenerate unitary polarity of a projective space $\mathrm{PG}(n, q^2)$. Hermitian varieties of $\mathrm{PG}(n, q^2)$ are projectively equivalent, as they can be reduced to the canonical form

$$X_0^{q+1} + \ldots + X_n^{q+1} = 0$$

by a non-singular linear transformation of $\mathrm{PG}(n, q^2)$. A *generator* of $\mathcal{U}_n$ is defined to be a projective subspace of maximum dimension lying on $\mathcal{U}_n$, namely of dimension $\lfloor \frac{1}{2}(n-1) \rfloor$. General results on Hermitian varieties are due to Segre [22]; see also [15], [14], [16]. Here, some basic facts from [16, Section 23.2] are recalled. Let μ_n denote the number of points on $\mathcal{U}_n$.

Result 2.1 (1) $\mu_n = (q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2 - 1)$.

(2) *For any point $P \in \mathcal{U}_n$, the number of lines through P and contained in $\mathcal{U}_n$ is equal to μ_{n-2}.*

(3) *The tangent hyperplane at $P \in \mathcal{U}_n$ meets $\mathcal{U}_n$ in $q^2\mu_{n-2} + 1$ points.*

Now we give a lower bound for the size of complete caps which does not depend on n.

Theorem 2.2 *The size k of a complete cap of $\mathcal{U}_n$ satisfies $k \geq q^2 + 1$.*

Proof The assertion is true for ovoids. Let $\mathcal{K}$ be a complete cap of $\mathcal{U}_n$ that is not an ovoid. Take a generator H of $\mathcal{U}_n$ disjoint from $\mathcal{K}$. For any point $P \in \mathcal{K}$, the tangent hyperplane Π_P to $\mathcal{U}_n$ at P does not contain H. In fact, some point of H is not conjugate to P, as H is a projective subspace of maximum dimension contained in $\mathcal{U}_n$. This implies that $\Pi_P \cap H$ is a hyperplane $H(P)$ of H. As $\mathcal{K}$ is a complete cap of $\mathcal{U}_n$, the projective subspaces $H(P)$ cover H as P ranges over $\mathcal{K}$. Since H is a projective space of dimension $r = \lfloor \frac{1}{2}(n-1) \rfloor$, this yields

$$1 + q^2 + \ldots + q^{2r} \leq k(1 + q^2 + \ldots + q^{2(r-1)}).$$

Hence

$$k \geq q^2 + 1/(1 + q^2 + \ldots + q^{2(r-1)}).$$

Since k is an integer, this is only possible for $k \geq q^2 + 1$. □

The above lower bound is sharp for $n = 3$ and even q; see Example 3.6 and Theorem 4.1 with $g = 0$. For the classification of transitive ovoids when $n = 3$ and q is even, see [5]. It is not known whether the lower bound remains true for $n > 3$ or for $n = 3$ and arbitrary odd q. To the best of our knowledge, the smallest complete cap of $\mathcal{U}_n$ for q is that described in the following theorem.

Theorem 2.3 *Let α be a plane of $\mathrm{PG}(n, q^2)$ which meets $\mathcal{U}_n$ in a non-degenerate Hermitian curve $\mathcal{U}_2$. Then $\mathcal{U}_2$ is a complete cap of $\mathcal{U}_n$ of size $q^3 + 1$.*

Proof First, $\mathcal{U}_2$ is a cap of $\mathcal{U}_n$. Let $A \in \mathcal{U}_n$ be any point. The tangent hyperplane Π_A to $\mathcal{U}$ at A either contains α or meets it in a line ℓ. It turns out in both cases that Π_A has a common point with $\mathcal{U}_2$, whence the assertion follows. □

3 Hermitian varieties and maximal curves

In algebraic geometry in positive characteristic the Hermitian variety is defined to be the hypersurface $\overline{\mathcal{U}}_n$ of homogeneous equation

$$X_0^{q+1} + \ldots + X_n^{q+1} = 0,$$

viewed as an algebraic variety in $\mathrm{PG}(n, \overline{\mathbf{F}})$ where $\overline{\mathbf{F}}$ is the algebraic closure of $\mathbf{F}_{q^2}$. Points of $\mathcal{U}_n$ are the points of $\overline{\mathcal{U}}_n$ with coordinates in $\mathbf{F}_{q^2}$, usually called $\mathbf{F}_{q^2}$-rational points of $\overline{\mathcal{U}}_n$. For a point $A = (a_0, a_1, \ldots, a_n)$ of $\overline{\mathcal{U}}_n$, the tangent hyperplane to $\overline{\mathcal{U}}_n$ at A has equation

$$a_0^q X_0 + a_1^q X_1 + \ldots + a_n^q X_n = 0.$$

In this paper, the term *algebraic curve defined over* $\mathbf{F}_{q^2}$ stands for a projective, geometrically irreducible, non-singular algebraic curve $\mathcal{X}$ of $\mathrm{PG}(n, q^2)$ viewed as a curve of $\mathrm{PG}(n, \overline{\mathbf{F}})$. Further, $\mathcal{X}(\mathbf{F}_{q^{2i}})$ denotes the set of points of $\mathcal{X}$ with all coordinates in $\mathbf{F}_{q^{2i}}$, called $\mathbf{F}_{q^{2i}}$-rational points of $\mathcal{X}$. For a point $P = (x_0, \ldots, x_n)$ of $\mathcal{X}$, the Frobenius image of P is defined to be the point $\Phi(P) = (x_0^{q^2}, \ldots, x_n^{q^2})$. Then $P = \Phi(P)$ if and only if $P \in \mathcal{X}(\mathbf{F}_{q^2})$.

An algebraic curve $\mathcal{X}$ defined over $\mathbf{F}_{q^2}$ is called $\mathbf{F}_{q^2}$*-maximal* if the number N_{q^2} of its $\mathbf{F}_{q^2}$-rational points attains the Hasse–Weil upper bound, namely $N_{q^2} = q^2 + 1 + 2gq$, where g denotes the genus of $\mathcal{X}$. In recent years, $\mathbf{F}_{q^2}$-maximal curves have been the subject of numerous papers; a motivation for their study comes from coding theory based on algebraic curves having many points over a finite field. Here, only results on maximal curves which play a role in the present investigation are gathered.

Result 3.1 (Natural embedding theorem [17]) *Up to* $\mathbf{F}_{q^2}$*-isomorphism, the* $\mathbf{F}_{q^2}$*-maximal curves of* $\mathrm{PG}(n, q^2)$ *are the algebraic curves defined over* $\mathbf{F}_{q^2}$ *of degree* $q + 1$ *and contained in the non-degenerate Hermitian variety* $\overline{\mathcal{U}}_n$*.*

Remark 3.2 The $\mathbf{F}_{q^2}$-maximality of $\mathcal{X}$ implies that $(q+1)P \equiv qQ + \Phi(Q)$ for every $Q \in \mathcal{X}$, and the natural embedding arises from the smallest linear series Σ containing all such divisors. Apart from some exceptions, Σ is complete and hence $\Sigma = |(q+1)P_0|$ for any $P_0 \in \mathbf{F}_{q^2}$. By the Riemann–Roch theorem, $\dim \Sigma = q + 1 - g + i$ where i is the index of speciality. In many situations, for instance when $q+1 > 2g-2$, we have $i = 0$, and hence $\dim \Sigma = q+1-g$. With our notation, $n = \dim \Sigma$.

This, together with some more results from [17], gives the following.

Result 3.3 *Let* $\mathcal{X}$ *be an* $\mathbf{F}_{q^2}$*-maximal curve naturally embedded in* $\overline{\mathcal{U}}_n$*. For a point* $P \in \mathcal{X}$*, let* Π_P *be the tangent hyperplane to* $\overline{\mathcal{U}}_n$ *at* P*. Then* Π_P *coincides with the hyper-osculating hyperplane to* $\mathcal{X}$ *at* P*, and*

$$\Pi_P \cap \mathcal{X} = \begin{cases} \{P\} & \text{for } P \in \mathcal{X}(\mathbf{F}_{q^2}), \\ \{P, \Phi(P)\} & \text{for } P \in \mathcal{X} \backslash \mathcal{X}(\mathbf{F}_{q^2}). \end{cases} \tag{3.1}$$

More precisely, for the intersection divisor D cut out on $\mathcal{X}$ by Π_P,

$$D = \begin{cases} (q+1)P & \text{for } P \in \mathcal{X}(\mathbf{F}_{q^2}), \\ qP + \Phi(P) & \text{for } P \in \mathcal{X}\backslash\mathcal{X}(\mathbf{F}_{q^2}). \end{cases} \tag{3.2}$$

Theorem 3.4 *Let $\mathcal{X}$ be an $\mathbf{F}_{q^2}$-maximal curve naturally embedded in $\overline{\mathcal{U}}_n$. For a point $A \in \mathcal{U}_n\backslash\mathcal{X}$, let Π_A be the tangent hyperplane to $\mathcal{U}_n$ at A. If $n = 3$ and q is even, then Π_A has a common point with $\mathcal{X}(\mathbf{F}_{q^2})$.*

Proof Let ℓ be a line of $\mathcal{U}_n$. Then ℓ, viewed as a line of $\mathrm{PG}(n, \overline{\mathbf{F}})$, is contained in $\overline{\mathcal{U}}_n$. Let $Q \in \ell \cap \mathcal{X}$; then it must be shown that $Q \in \mathcal{X}(\mathbf{F}_{q^4})$.

Assume, on the contrary, that $Q \in \mathcal{X}(\mathbf{F}_{q^{2i}})$ with $i \geq 3$. Then the three points $Q, \Phi(Q), \Phi(\Phi(Q))$ are distinct points of $\mathcal{X}$. Since ℓ is defined over $\mathbf{F}_{q^2}$, so ℓ contains not only Q but also $\Phi(Q)$ and $\Phi(\Phi(Q))$. By (3.1), the hyperosculating hyperplane Π_Q to $\mathcal{X}$ at Q contains $\Phi(Q)$, and hence Π_Q contains the line ℓ. But then Π_Q must contain $\Phi(\Phi(Q))$, contradicting (3.1).

Assume now that $Q \in \mathcal{X}(\mathbf{F}_{q^4})$. The previous argument also shows that ℓ contains both Q and $\Phi(Q)$ but no more points from $\mathcal{X}$. Also, ℓ cannot contain more than one point from $\mathcal{X}(\mathbf{F}_{q^2})$, again by (3.1). Hence, if $\ell \cap \mathcal{X}$ is non–trivial, then either $\ell \cap \mathcal{X}$ is a single $\mathbf{F}_{q^2}$-rational point or $\ell \cap \mathcal{X}$ consists of two distinct points, Frobenius images of each other, both in $\mathcal{X}(\mathbf{F}_{q^4})\backslash\mathcal{X}(\mathbf{F}_{q^2})$.

Let $Q \in \overline{\mathcal{U}}_n$ be any point in $\Pi_A \cap \mathcal{X}$. Then the line ℓ through A and Q is contained in $\overline{\mathcal{U}}_n$. Now, assume that $n = 3$; then such a line is contained in $\mathcal{U}_n$. By the above assertions, the points in $\Pi_A \cap \mathcal{X}$ are $\mathbf{F}_{q^4}$-rational points of $\mathcal{X}$. For a point $Q \in \mathcal{X}$, let $I(\mathcal{X}, \Pi_A; Q)$ denote the intersection multiplicity of $\mathcal{X}$ and Π_A at Q. By Bézout's theorem, $\sum_Q I(\mathcal{X}, \Pi_A; Q) = q + 1$ where Q ranges over all points of $\mathcal{X}$. Write

$$\sum_Q I(\mathcal{X}, \Pi_A; Q) = \sum_Q{}' I(\mathcal{X}, \Pi_A; Q) + \sum_Q{}'' I(\mathcal{X}, \Pi_A; Q),$$

where the summation $\sum'$ is over $\mathcal{X}(\mathbf{F}_{q^2})$ while $\sum''$ is over $\mathcal{X}(\mathbf{F}_{q^4})\backslash\mathcal{X}(\mathbf{F}_{q^2})$. Since both Π_A and $\mathcal{X}$ are defined over $\mathbf{F}_{q^2}$,

$$I(\mathcal{X}, \Pi_A; Q) = I(\mathcal{X}, \Pi_A; \Phi(Q)).$$

Hence $\sum_Q'' I(\mathcal{X}, \Pi_A; Q) \equiv q + 1 \pmod 2$. For q even, this implies that $I(\mathcal{X}, \Pi_A; Q) > 0$ for at least one point $Q \in \mathcal{X}(\mathbf{F}_{q^2})$, whence the assertion follows. □

Remark 3.5 Theorem 3.4 might not extend to $n > 3$. For a point $A \in \mathcal{U}_n$, let $Q \in \overline{\mathcal{U}}_n$ be a point other than A in the tangent hyperplane Π_A of $\mathcal{U}_n$ at A. If $n = 3$, then the line ℓ through A and Q is $\mathbf{F}_{q^2}$-rational. But this assertion does not hold true for $n > 3$.

In fact, let $\mathcal{U}_n$ be given in its canonical form

$$X_0^q X_n + X_0 X_n^q + X_1^{q+1} + \ldots + X_{n-1}^{q+1} = 0.$$

It may be assumed that $A = (0, \ldots, 0, 1)$. Then Π_A has equation $X_0 = 0$ and $Q = (0, a_1, \ldots, a_{n-1}, 1)$ with $a_1^{q+1} + \ldots + a_{n-1}^{q+1} = 0$. The line ℓ is $\mathbf{F}_{q^2}$-rational if and only if $\Phi(Q)$ also lies on ℓ. This happens when $a_i^{q^2} = \lambda a_i$, $i = 1, \ldots, n-1$, for a suitable element $\lambda \in \overline{\mathbf{F}}$, or, equivalently, when $a_i^{q^2-1} = a_j^{q^2-1}$ for all i, j with $1 \leq i, j \leq n-1$ and $a_i, a_j \neq 0$. Now, $a_1^{q+1} = -a_2^{q+1}$ implies $(a_1^{q+1})^{q-1} = (a_2^{q+1})^{q-1}$, whence the assertion follows for $n = 3$. Unfortunately, as soon as $n > 3$, $a_1^{q+1} + \ldots + a_{n-1}^{q+1} = 0$ does not imply $a_i^{q^2-1} = a_j^{q^2-1}$ for any i, j with $1 \leq i, j \leq n-1$ and $a_i, a_j \neq 0$. Thus the assertion is not valid for $n > 3$.

The following example illustrates property (3.1).

Example 3.6 Still with q even, write the equation of $\mathcal{U}_3$ in the form

$$X_0^q X_3 + X_0 X_3^q = X_1^{q+1} + X_2^{q+1}.$$

The rational algebraic curve $\mathcal{X}$ of degree $q+1$, consisting of all points

$$A(t) = \{(1, t, t^q, t^{q+1}) \mid t \in \overline{\mathbf{F}}\}$$

together with the point $A(\infty) = (0, 0, 0, 1)$, lies on $\mathcal{U}_3$. The morphism

$$(1, t) \to (1, t, t^q, t^{q+1})$$

is a natural embedding. We note that the tangent hyperplane $\Pi_{A(t)}$ to $\overline{\mathcal{U}}_3$ at $A(t)$ has equation

$$t^{q(q+1)} X_0 + X_3 + t^q X_1 + t^{q^2} X_2 = 0.$$

To show that (3.1) holds for $A(t)$, it is necessary to check that the equation

$$t^{q(q+1)} + u^{q+1} + t^q u + t^{q^2} u^q = 0$$

has only two solutions in u, namely $u = t$ and $u = t^{q^2}$. Replacing u by $v + t$, the equation becomes $v^{q+1} + v^q t + t^{q^2} v^q = 0$. For $v \neq 0$, that is, for $u \neq t$, this implies $v = t^{q^2} + t$, proving the assertion. For $A(\infty)$, the tangent hyperplane $\Pi_{A(\infty)}$ has equation $X_0 = 0$. Hence it does not meet $\mathcal{X}$ outside $A(\infty)$, showing that (3.1) also holds for $A(\infty)$.

4 Caps of the Hermitian variety arising from maximal curves

From the results stated in Section 3 we deduce the following theorem.

Theorem 4.1 *Let $\mathcal{X}$ be an $\mathbf{F}_{q^2}$-maximal curve naturally embedded in $\overline{\mathcal{U}}_n$. Then*

(i) *$\mathcal{X}(\mathbf{F}_{q^2})$ is a cap of $\mathcal{U}_n$ of size $q^2+1+2gq$;*

(ii) *when q is even and $n=3$, such a cap is complete.*

Proof Let $P \in \mathcal{X}(\mathbf{F}_{q^2})$. By (3.1), no further point from $\mathcal{X}$ is in Π_P. Hence no point in $\mathcal{X}(\mathbf{F}_{q^2})$ is conjugate to P. This shows that $\mathcal{X}(\mathbf{F}_{q^2})$ is a cap of $\mathcal{U}_n$ whose size is equal to $q^2+1+2gq$ by the $\mathbf{F}_{q^2}$-maximality of $\mathcal{X}$. Completeness for even q and $n=3$ follows from Theorem 3.4. □

In applying Theorem 4.1 it is essential to have information on the spectrum of the genera g of $\mathbf{F}_{q^2}$-maximal curves. However, it would be inappropriate in the present paper to discuss the spectrum in all details; so we shall content ourselves with a summary of the relevant results in characteristic 2. For this reason, q will denote a power of 2 in the rest of the paper, apart from Example 4.8.

Result 4.2 (1) *The lower limit of the spectrum of genera is* 0, *which is only attained by rational algebraic curves.*

(2) *The upper limit of the spectrum is $\frac{1}{2}(q^2-q)$, which is only attained by the Hermitian curve over $\mathbf{F}_{q^2}$; see* [23, Proposition V.3.3].

Result 4.3 [1, 10, 18]

(1) *The second largest value in the spectrum of genera is $\frac{1}{4}(q^2-2q)$, which is only attained by Example* 4.5.

(2) *In the interval $[\frac{1}{8}(q^2-4q+3), \frac{1}{4}(q^2-q)]$, there are* 12 *known examples.*

Result 4.4 [18] *The third largest value in the spectrum is $\lfloor\frac{1}{6}(q^2-q+4)\rfloor$. Examples* 4.6 *and* 4.7 *are the only known examples with this genus.*

Example 4.5 ([9]). The absolutely irreducible plane curve $\mathcal{C}$ with equation

$$y + y^2 + \ldots + y^{q/2} + x^{q+1} = 0$$

has genus $\frac{1}{4}q(q-2)$. A non-singular model $\mathcal{X}$ of $\mathcal{C}$ is the $\mathbf{F}_{q^2}$-maximal curve defined by the morphism $\pi : \mathcal{C} \to \mathrm{PG}(3, q^2)$ with coordinate functions

$$f_0 = 1,\ f_1 = x,\ f_2 = y,\ f_3 = x^2.$$

The curve $\mathcal{X}$ lies on the Hermitian variety $\overline{\mathcal{U}}_3$ with equation

$$X_2^q X_0 + X_2 X_0^q + X_1^{q+1} + X_3^{q+1} = 0.$$

Also, $\mathcal{X}$ lies on the quadric cone with equation $X_3 X_0 = X_1^2$. The size of the corresponding complete cap $\mathcal{X}(\mathbf{F}_{q^2})$ of $\mathcal{U}_3$ is $\frac{1}{2}(q^3+2)$.

Example 4.6 ([7, Theorem 2.1.(IV)(2)]) Let $q \equiv 2 \pmod 3$. The absolutely irreducible plane curve $\mathcal{C}$ with equation $x^{(q+1)/3} + x^{2(q+1)/3} + y^{q+1} = 0$ has genus $g = \frac{1}{6}(q^2 - q + 4)$. A non-singular model $\mathcal{X}$ of $\mathcal{C}$ is the $\mathbf{F}_{q^2}$-maximal curve defined by the morphism $\pi : \mathcal{C} \to \mathrm{PG}(3, q^2)$ with coordinate functions

$$f_0 = x,\ f_1 = x^2,\ f_2 = y^3,\ f_3 = xy.$$

The curve $\mathcal{X}$ lies on the Hermitian variety $\overline{\mathcal{U}}_3$ given by the usual canonical equation

$$X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0.$$

Also, $\mathcal{X}$ lies on the cubic surface with equation

$$X_3^3 + w^3 X_0 X_1 X_2 = 0$$

with $w^{q+1} = -3$. The size of the corresponding complete cap $\mathcal{X}(\mathbf{F}_{q^2})$ of $\mathcal{U}_3$ is $\frac{1}{3}(q^3 + 2q^2 + 4q + 3)$.

Example 4.7 ([6, §6]) A similar but non-isomorphic example is given in [6]. Again, assume that $q \equiv 2 \pmod 3$. The absolutely irreducible plane curve $\mathcal{C}$ with equation

$$yx^{(q-2)/3} + y^q + x^{(2q-1)/3} = 0$$

has genus $\frac{1}{6}(q^2 - q - 2)$. A non-singular model $\mathcal{X}$ of $\mathcal{C}$ is the $\mathbf{F}_{q^2}$-maximal curve defined by the morphism $\pi : \mathcal{C} \to \mathrm{PG}(3, q^2)$ with coordinate functions

$$f_0 = x,\ f_1 = x^2,\ f_2 = y^3,\ f_3 = -3xy.$$

The curve $\mathcal{X}$ lies on the Hermitian variety Σ_{q+1} with equation

$$X_0^q X_1 + X_1^q X_2 + X_2^q X_0 - 3X_3^{q+1} = 0.$$

Also, $\mathcal{X}$ is contained in the cubic surface with equation

$$X_3^3 + 27X_0X_1X_2 = 0.$$

It is worth noting that Σ_{q+1} is projectively equivalent to $\mathcal{U}_3$ in $\mathrm{PG}(3, q^6)$ but not in $\mathrm{PG}(3, q^2)$. Nevertheless, the projective transformation taking Σ_{q+1} to $\mathcal{U}_3$ maps $\mathcal{X}$ to an $\mathbf{F}_{q^2}$-maximal curve lying on $\mathcal{U}_3$. The size of the corresponding complete cap $\mathcal{X}(\mathbf{F}_{q^2})$ of $\mathcal{U}_3$ is $\frac{1}{3}(q^3 + 2q^2 - 2q + 3)$.

We end the paper with an example for q odd which shows that assertion (ii) in Theorem 4.1 does not hold for q odd.

Example 4.8 Let q be odd and let $\mathcal{C}(\mathbf{F}_{q^2})$ be the absolutely irreducible plane curve with equation

$$y^q + y + x^{(q+1)/2} = 0;$$

it has genus $\frac{1}{4}(q-1)^2$. A non-singular model $\mathcal{X}$ of $\mathcal{C}$ is the $\mathbf{F}_{q^2}$-maximal curve defined by the morphism $\pi : \mathcal{C} \to \mathrm{PG}(3, q^2)$ with coordinate functions

$$f_0 = 1,\ f_1 = x,\ f_2 = y,\ f_3 = y^2.$$

The curve $\mathcal{X}$ lies on the Hermitian surface $\mathcal{U}_3$ with equation

$$X_3^q X_0 + X_3 X_0^q + 2X_2^{q+1} - X_1^{q+1} = 0.$$

Also, $\mathcal{C}$ lies on the quadric cone $\mathcal{Q}$ with equation $X_2^2 - X_0X_3 = 0$. The size of the corresponding cap $\mathcal{K}$ of $\mathcal{U}_3$ is $q^2 + 1 + \frac{1}{2}q(q-1)^2 = \frac{1}{2}(q^3 + q + 2)$. The cap $\mathcal{K}$ is incomplete, since it is contained in an ovoid of $\mathcal{U}_3$; see [13].

Acknowledgements

The second author's research was carried out within the project "Strutture geometriche, combinatoria e applicazioni" PRIN 2001-02, MIUR.

References

[1] M. Abdón and F. Torres, On maximal curves in characteristic two, *Manuscripta Math.* **99** (1999), 39–53.

[2] R.D. Baker, G.L. Ebert, G. Korchmáros and T. Szőnyi, Orthogonally divergent spreads of Hermitian curves, in *Finite Geometry and Combinatorics, F. De Clerck et al. (eds), London Math. Soc. Lecture Note Series* **191** (1993) 17–30.

[3] A.E. Brouwer and H. Wilbrink, Ovoids and fans in the generalized quadrangles $Q(4,2)$, *Geom. Dedicata* **36** (1990), 121–124.

[4] A. Cossidente, J.W.P. Hirschfeld, G. Korchmáros and F. Torres, On plane maximal curves *Compositio Math.* **121** (2000), 163–181.

[5] A. Cossidente and G. Korchmáros, Transitive ovoids of the Hermitian surface of $\mathrm{PG}(3,q^2)$, q even, *J. Combin. Theory Ser. A*, **101** (2003), 117–130.

[6] A. Cossidente, G. Korchmáros and F. Torres, On curves covered by the Hermitian curve, *J. Algebra* **216** (1999), 56–76.

[7] A. Cossidente, G. Korchmáros and F. Torres, Curves of large genus covered by the Hermitian curve, *Comm. Algebra* **28** (2000), 4707–4728.

[8] G. Faina, A characterization of the tangent lines to a Hermitian curve, *Rend. Mat.* **3** (1983), 553–557.

[9] R. Fuhrmann, A. Garcia and F. Torres, On maximal curves, *J. Number Theory* **67** (1997), 29-51.

[10] R. Fuhrmann and F. Torres, The genus of curves over finite fields with many rational points, *Manuscripta Math.* **89** (1996), 103–106.

[11] R. Fuhrmann and F. Torres, On Weierstrass points and optimal curves, *Rend. Circ. Mat. Palermo* **51** (1998), 25–46.

[12] A. Garcia and J.F. Voloch, Fermat curves over finite fields, *J. Number Theory* **30** (1988), 345-356.

[13] L. Giuzzi and G. Korchmáros, Ovoids of the Hermitian surface in odd characteristic, *Adv. Geom.*, to appear.

[14] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.

[15] J.W.P. Hirschfeld, *Projective Geometries Over Finite Fields*, second edition, Oxford University Press, Oxford, 1998.

[16] J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries*, Oxford University Press, Oxford, 1991.

[17] G. Korchmáros and F. Torres, Maximal curves embedded in a Hermitian variety, *Compositio Math.* **128** (2001), 95–113.

[18] G. Korchmáros and F. Torres, On the genus of a maximal curve, *Math. Ann.* **323**, (2002), 589–608.

[19] G.E. Moorhouse, Some p-ranks related to Hermitian varieties, *J. Statist. Plann. Inference* **56** (1996), 229–241.

[20] S.E. Payne and J.A. Thas, Spreads and ovoids in finite generalized quadrangles, *Geom. Dedicata* **52** (1994), 227–253.

[21] H.G. Rück and H. Stichtenoth, A characterization of Hermitian function fields over finite fields, *J. Reine Angew. Math.* **457** (1994), 185–188.

[22] B. Segre, Forme e geometrie hermitiane, con particolare riguardo al caso finito, *Ann. Mat. Pura Appl.* **70** (1965), 1–201.

[23] H. Stichtenoth, *Algebraic Function Fields and Codes*, Springer-Verlag, Berlin, 1993.

[24] J.A. Thas, Ovoids and spreads of finite classical polar spaces, *Geom. Dedicata* **10**, 135–144.

[25] J.A. Thas, Old and new results on spreads and ovoids of finite classical polar spaces, in A. Barlotti et al. (eds) *Combinatorics'90, Ann. Discrete Math.* **52** (1992), 529–544.

[26] G. van der Geer and M. van der Vlugt, Tables of curves with many points, *Math. Comp.* **69** (2000), 797–810;
`http://www.wins.uva.nl/~geer`

School of Mathematical Sciences
University of Sussex
Brighton BN1 9QH
United Kingdom
jwph@sussex.ac.uk

Dipartimento di Matematica
Università della Basilicata
85100 Potenza
Italy
korchmaros@unibas.it