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Invited Comment

Quantum entanglement for systems of identical bosons: II. Spin squeezing and other entanglement tests

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Abstract

These two accompanying papers are concerned with entanglement for systems of identical massive bosons and the relationship to spin squeezing and other quantum correlation effects. The main focus is on two mode entanglement, but multi-mode entanglement is also considered. The bosons may be atoms or molecules as in cold quantum gases. The previous paper I dealt with the general features of quantum entanglement and its specific definition in the case of systems of identical bosons. Entanglement is a property shared between two (or more) quantum sub-systems. In defining entanglement for systems of identical massive particles, it was concluded that the single particle states or modes are the most appropriate choice for sub-systems that are distinguishable, that the general quantum states must comply both with the symmetrization principle and the super-selection rules (SSR) that forbid quantum superpositions of states with differing total particle number (global SSR compliance). Further, it was concluded that (in the separable states) quantum superpositions of sub-system states with differing sub-system particle number (local SSR compliance) also do not occur. The present paper II determines possible tests for entanglement based on the treatment of entanglement set out in paper I. Several inequalities involving variances and mean values of operators have been previously proposed as tests for entanglement between two sub-systems. These inequalities generally involve mode annihilation and creation operators and include the inequalities that define spin squeezing. In this paper, spin squeezing criteria for two mode systems are examined, and spin squeezing is also considered for principle spin operator components where the covariance matrix is diagonal. The proof, which is based on our SSR compliant approach shows that the presence of spin squeezing in any one of the spin components requires entanglement of the relevant pair of modes. A simple Bloch vector test for entanglement is also derived. Thus we show that spin squeezing becomes a rigorous test for entanglement in a system of massive bosons, when viewed as a test for entanglement between two modes. In addition, other previously proposed tests for entanglement involving spin operators are considered, including those based on the sum of the variances for two spin components. All of the tests are still valid when the present concept of entanglement based on the symmetrization and SSR criteria is applied. These tests also apply in cases of multi-mode entanglement, though with restrictions in the case of sub-systems each consisting of pairs of modes. Tests involving quantum correlation functions are also considered and for global SSR compliant states these are shown to be equivalent to tests involving spin operators. A new weak correlation test is derived for entanglement based on local SSR compliance for separable states, complementing the stronger correlation test obtained previously when this is ignored. The Bloch vector test is equivalent to one case of this weak correlation test. Quadrature squeezing for single modes is also examined but not found to yield a useful entanglement test, whereas two mode quadrature squeezing proves to be a valid entanglement test, though not as useful as the Bloch vector test. The various entanglement tests are considered for well-known entangled states, such as binomial states, relative phase eigenstates and NOON states—sometimes the new tests are satisfied while than those obtained in other papers are not. The present paper II then outlines the theory for a simple two mode interferometer showing that such an interferometer can be used to measure the mean values and covariance matrix for the spin operators involved in entanglement tests for the two mode bosonic system. The treatment is also generalized to cover multi-mode interferometry. The interferometer involves a pulsed classical field characterized by a phase variable and an area variable defined by the time integral of the field amplitude, and leads to a coupling between the two modes. For simplicity the center frequency was chosen to be resonant with the inter-mode transition frequency. Measuring the mean and variance of the population difference between the two modes for the output state of the interferometer for various choices of interferometer variables is shown to enable the mean values and covariance matrix for the spin operators for the input quantum state of the two mode system to be determined. The paper concludes with a discussion of several key experimental papers on spin squeezing.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The previous paper [1] I dealt with the general features of quantum entanglement and its specific definition in the case of systems of identical massive bosons. In defining entanglement for systems of identical massive particles, it was concluded that the single particle states or modes are the most appropriate choice for sub-systems that are distinguishable. Further, it was concluded that the general quantum states must comply both with the symmetrization principle and the super-selection rules (SSR) forbidding quantum superpositions of states with differing total particle number (global SSR compliance). As a consequence, it was then reasoned that in the separable states quantum superpositions of sub-system states with differing sub-system particle number (local SSR compliance) do not occur [2]. Other approaches—such as sub-systems consisting of labeled indistinguishable particles and entanglement due to symmetrization [3] or allowing for non-entangled separable but non-local states [4]—were found to be unsuitable. The local (and global) SSR compliant definition of entanglement used here was justified on the basis of there being no non-relativistic quantum processes available to create SSR non-compliant states and alternatively on the absence of a phase reference [5].

Paper I can be summarized as follows. Section 2 covered the key definitions of entangled states, the relationship to hidden variable theory and some of the key paradoxes associated with quantum entanglement such as EPR and Bell inequalities. Details were set out in appendices A, B, D and G. Conditional probabilities relevant to EPR inequalities re treated in appendix C. Details about the position/momentum and spin EPR paradoxes set out in appendix F. A detailed discussion on why the symmetrization principle and the SSR is invoked for entanglement in identical particle systems was discussed in section 3 with details in appendix N. Challenges to the necessity of the SSR were outlined, with arguments against such challenges dealt with in appendices I, K and M. The applicability (or otherwise) of the SSRs for photons was discussed in appendix L. Results about quantum correlation functions were presented in appendices H and J. Two key mathematical inequalities were derived in appendix E. The final section 4 summarized the key features of quantum entanglement discussed in paper I. The appendices are available as online supplementary material, stacks.iop.org/ps/92/023005/mmedia. References [43–48] are discussed therein.

The present paper II focuses on tests for entanglement in two mode systems of identical bosons, with particular emphasis on spin squeezing and correlation tests and how the quantities involved in these tests can be measured via two mode interferometry. Two mode bosonic systems are of particular interest because cold atomic gases cooled well below the Bose–Einstein condensation (BEC) transition temperature can be prepared where essentially only two modes are occupied [6, 7]. This can be achieved for cases involving a single hyperfine components using a double well trap potential or for two hyperfine components using a single well. At higher temperatures more than two modes may be occupied, so multi-mode systems are also of importance and the two mode treatment is extended to this situation.

As well as their relevance for entanglement tests, states that are spin squeezed have important applications in *quantum metrology*. That squeezed states can improve interferometry via the quantum noise in quadrature variables being reduced to below the standard quantum limit has been known since the pioneering work of Caves [8] on optical systems. The extension to spin squeezing in systems of massive bosons originates with the work of Kitagawa and Ueda [9], who considered systems of two state atoms. As this review is focused on spin squeezing as an entanglement test rather than the use of spin squeezing in quantum metrology, the latter subject will not be covered here. In quantum metrology involving spin operators the quantity $\sqrt{\langle\Delta\widehat{S}_x^2\rangle}/|\langle\widehat{S}_z\rangle|$ (which involves the variance and mean value of orthogonal spin operators) is a measure of the uncertainty $\Delta\theta$ in measuring the interferometer *phase*. The interest in spin squeezing lies in the possibility of improvement over the *standard quantum limit* where $\Delta\theta = 1/\sqrt{N}$ (see section 3.6). As we will see, for squeezed states $\sqrt{\langle\Delta\widehat{S}_x^2\rangle} < \sqrt{|\langle\widehat{S}_z\rangle|/2}$, so we could have $\Delta\theta < \sim 1/\sqrt{|\langle\widehat{S}_z\rangle|} \sim 1/\sqrt{N}$ which is less than the standard quantum limit. In section 3.7 we give an example of a highly squeezed state where $\Delta\theta \sim \sqrt{\ln N}/N$ which is near the *Heisenberg limit*. Suffice to say that increasing the number of particles in the squeezed state has the effect of improving the sensitivity of the interferometer. Aspects of quantum metrology are covered in a number of papers (see [10, 11]), based on concepts such as quantum Fisher information, Cramers–Rao bound [12, 13], quantum phase eigenstates.

The proof of the key conclusion that spin squeezing in any spin component is a sufficiency test for entanglement [2] is set out in this paper, as is that for a new Bloch vector test. A previous proof [14] that spin squeezing in the z spin component $\sqrt{\langle\Delta\widehat{S}_z^2\rangle} < \sqrt{|\langle\widehat{S}_x\rangle|/2}$ demonstrates entanglement based on treating identical bosonic atoms as distinguishable sub-systems has therefore now been superseded. It is seen that correlation tests for entanglement of quantum states complying with the global particle number SSR can be expressed in terms of inequalities involving powers of spin operators. Section 2 sets out the definitions of spin squeezing and in the following section 3 it is shown that spin squeezing is a test for entanglement, both for the original spin operators with entanglement of the original modes, for the principle spin operators with entanglement of the two new modes and finally for several multi-mode cases. Details of the latter are set out in appendices A and D. A number of other correlation, spin operator and quadrature operator tests for entanglement proposed by other authors are considered in sections 4–6, with details of these treatments set out in appendices G and J. Some tests also apply in cases of multi-mode entanglement, though with restrictions in the case of sub-systems each consisting of pairs of modes. A new weak correlation test is derived and for one case is equivalent to the Bloch vector test.

In section 7 it is shown that a simple two mode interferometer can be used to measure the mean values and covariance matrix for the spin operators involved in entanglement tests, with details covered in appendices M and N. The treatment is also generalized to cover multi-mode interferometry. Actual experiments aimed at detecting entanglement via spin squeezing tests are examined in section 8. The final section 9 summarizes and discusses the key results regarding entanglement tests. Appendices K and O provide details regarding certain important states whose features are discussed in this paper—the 'separable but non-local' states and the relative phase eigenstate. The appendices are available as online supplementary material, stacks.iop.org/ps/92/023005/mmedia. References [43–48] are cited therein.

2. Spin squeezing

The basic concept of spin squeezing was first introduced by Kitagawa and Ueda [9] for general spin systems. These include cases based on two mode systems, such as may occur both for optical fields and for Bose–Einstein condensates. Though focused on systems of massive identical bosons, the treatment in this paper also applies to photons though details will differ.

2.1. Spin operators, Bloch vector and covariance matrix

2.1.1. Spin operators. For two mode systems with mode annihilation operators $\widehat{a}$, $\widehat{b}$ associated with the two single particle states $|\phi_a\rangle$, $|\phi_b\rangle$, and where the non-zero bosonic commutation rules are $[\widehat{e}, \widehat{e}^\dagger] = \widehat{1}$ ($\widehat{e} = \widehat{a}$ or $\widehat{b}$), Schwinger *spin angular momentum operators* $\widehat{S}_\xi$ $(\xi = x, y, z)$ are defined as

$$\widehat{S}_x = (\widehat{b}^\dagger\widehat{a} + \widehat{a}^\dagger\widehat{b})/2, \qquad \widehat{S}_y = (\widehat{b}^\dagger\widehat{a} - \widehat{a}^\dagger\widehat{b})/2i,$$
$$\widehat{S}_z = (\widehat{b}^\dagger\widehat{b} - \widehat{a}^\dagger\widehat{a})/2 \qquad (1)$$

and which satisfy the commutation rules $[\widehat{S}_\xi, \widehat{S}_\mu] = i\epsilon_{\xi\mu\lambda}\widehat{S}_\lambda$ for angular momentum operators. For bosons the square of the angular momentum operators is given by $\widehat{S}_x^2 + \widehat{S}_y^2 + \widehat{S}_z^2 = (\widehat{N}/2)(\widehat{N}/2 + 1)$, where $\widehat{N} = (\widehat{b}^\dagger\widehat{b} + \widehat{a}^\dagger\widehat{a})$ is the boson total number operator, those for the separate modes being $\widehat{n}_e = \widehat{e}^\dagger\widehat{e}$ ($\widehat{e} = \widehat{a}$ or $\widehat{b}$). The Schwinger spin operators are the second quantization form of symmetrized one body operators $\widehat{S}_x = \sum_i(|\phi_b(i)\rangle\langle\phi_a(i)| + |\phi_a(i)\rangle\langle\phi_b(i)|)/2$; $\widehat{S}_y = \sum_i(|\phi_b(i)\rangle\langle\phi_a(i)| - |\phi_a(i)\rangle\langle\phi_b(i)|)/2i$; $\widehat{S}_z = \sum_i(|\phi_b(i)\rangle\langle\phi_b(i)| - |\phi_a(i)\rangle\langle\phi_a(i)|)/2$, where the sum i is over the identical bosonic particles. In the case of the two mode EM field the spin angular momentum operators are related to the Stokes parameters.

As well as spin operators for the simple case of two modes we can also define spin operators in multimode cases involving two sub-systems A and B. For example, there may be two types of bosonic particle involved, each component distinguished from the other by having different hyperfine internal states $|A\rangle$, $|B\rangle$. Each component may be associated with a complete orthonormal set of spatial mode functions $\phi_{ai}(r)$ and $\phi_{bi}(r)$, so there will be two sets of modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$, where in the $|\mathbf{r}\rangle$ representation we have $\langle \mathbf{r}\,|\phi_{ai}\rangle = \phi_{ai}(r)|A\rangle$ and $\langle \mathbf{r}\,|\phi_{bi}\rangle = \phi_{bi}(r)|B\rangle$. Mode orthogonality between A and B modes arises from $\langle A|B\rangle = 0$ rather from the spatial mode functions being orthogonal. The multimode spin operators are defined in appendix A (see equations (193) and (196)). These satisfy the standard commutation rules for angular momentum operators.

2.1.2. Bloch vector and covariance matrix. If the density operator for the overall system is $\hat{\rho}$ then expectation values of the three spin operators $\langle \hat{S}_\xi \rangle = \mathrm{Tr}(\hat{\rho}\hat{S}_\xi)$ $(\xi = x, y, z)$ define the *Bloch vector*. Spin squeezing is related to the fluctuation operators $\Delta\hat{S}_\xi = \hat{S}_\xi - \langle \hat{S}_\xi \rangle$, in terms of which a real, symmetric *covariance matrix* $C(\hat{S}_\xi, \hat{S}_\mu)$ $(\xi, \mu = x, y, z)$ is defined [7, 15] via

$$\begin{aligned} C(\hat{S}_\xi, \hat{S}_\mu) &= (\langle \Delta\hat{S}_\xi\, \Delta\hat{S}_\mu \rangle + \langle \Delta\hat{S}_\mu\, \Delta\hat{S}_\xi \rangle)/2 \\ &= \langle \hat{S}_\xi\, \hat{S}_\mu + \hat{S}_\mu\, \hat{S}_\xi \rangle/2 - \langle \hat{S}_\xi \rangle \langle \hat{S}_\mu \rangle \end{aligned} \tag{2}$$

and whose diagonal elements $C(\hat{S}_\xi, \hat{S}_\xi) = \langle \Delta\hat{S}_\xi^2 \rangle$ gives the variance for the fluctuation operators. The covariance matrix is also *positive definite*. The variances for the spin operators satisfy the three Heisenberg uncertainty principle relations $\langle \Delta\hat{S}_x^2 \rangle \langle \Delta\hat{S}_y^2 \rangle \geqslant \frac{1}{4} |\langle \hat{S}_z \rangle|^2$; $\langle \Delta\hat{S}_y^2 \rangle \langle \Delta\hat{S}_z^2 \rangle \geqslant \frac{1}{4} |\langle \hat{S}_x \rangle|^2$; $\langle \Delta\hat{S}_z^2 \rangle \langle \Delta\hat{S}_x^2 \rangle \geqslant \frac{1}{4} |\langle \hat{S}_y \rangle|^2$, and spin squeezing is defined via conditions such as $\langle \Delta\hat{S}_x^2 \rangle < \frac{1}{2} |\langle \hat{S}_z \rangle|$ with $\langle \Delta\hat{S}_y^2 \rangle > \frac{1}{2} |\langle \hat{S}_z \rangle|$, for $\hat{S}_x$ being squeezed compared to $\hat{S}_y$ and so on. Spin squeezing in these components is relevant to tests for entanglement of the modes $\hat{a}$ and $\hat{b}$, as will be shown later. Spin squeezing in rotated components is also important, in particular in the so-called *principal* components for which the covariance matrix is diagonal.

2.2. New spin operators and principal spin fluctuations

The covariance matrix has real, non-negative eigenvalues and can be diagonalized via an orthogonal *rotation matrix* $M(-\alpha, -\beta, -\gamma)$ that defines *new spin angular momentum operators* $\hat{J}_\xi$ $(\xi = x, y, z)$ via

$$\hat{J}_\xi = \sum_\mu M_{\xi\mu}(-\alpha, -\beta, -\gamma)\hat{S}_\mu \tag{3}$$

and where

$$\begin{aligned} C(\hat{J}_\xi, \hat{J}_\mu) &= \sum_{\lambda\theta} M_{\xi\lambda}(-\alpha, -\beta, -\gamma) C(\hat{S}_\lambda, \hat{S}_\theta) \\ &\quad \times M_{\mu\theta}(-\alpha, -\beta, -\gamma) \\ &= \delta_{\xi\mu} \langle \Delta\hat{J}_\xi^2 \rangle \end{aligned} \tag{4}$$

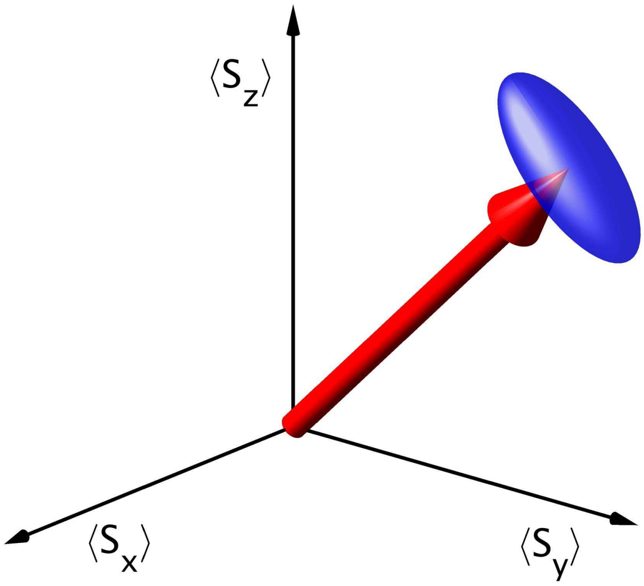

Figure 1. Bloch vector and spin fluctuations shown for original spin operators. The Bloch vector is shown via the red arrow and has components $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$, $\langle \hat{S}_z \rangle$. The blue ellipsoid represents the principal spin fluctations $\sqrt{\langle \Delta\hat{J}_x^2 \rangle}$, $\sqrt{\langle \Delta\hat{J}_y^2 \rangle}$ and $\sqrt{\langle \Delta\hat{J}_z^2 \rangle}$.

is the covariance matrix for the new spin angular momentum operators $\hat{J}_\xi$ $(\xi = x, y, z)$, and which is *diagonal* with the diagonal elements $\langle \Delta\hat{J}_x^2 \rangle$, $\langle \Delta\hat{J}_y^2 \rangle$ and $\langle \Delta\hat{J}_z^2 \rangle$ giving the so-called *principal spin fluctuations*. The matrix $M(\alpha, \beta, \gamma)$ is parameterized in terms of three Euler angles α, β, γ and is given in [16] (see equation (4.43)).

The Bloch vector and spin fluctuations are illustrated in figure 1. In figure 1 the Bloch vector and spin fluctuation ellipsoid is shown in terms of the original spin operators $\hat{S}_\xi$ $(\xi = x, y, z)$.

These rules also apply to multimode spin operators as defined in appendix A.

2.3. Spin squeezing definitions

We will begin by considering the case of the spin operators in the most general case. We will also specifically consider spin squeezing for the two new modes. Other cases are discussed in appendix B.

2.3.1. Heisenberg uncertainty principle and spin squeezing. Since the spin operators also satisfy *Heisenberg uncertainty principle* relationships

$$\begin{aligned} \langle \Delta\hat{S}_x^2 \rangle \langle \Delta\hat{S}_y^2 \rangle &\geqslant \frac{1}{4} |\langle \hat{S}_z \rangle|^2, \\ \langle \Delta\hat{S}_y^2 \rangle \langle \Delta\hat{S}_z^2 \rangle &\geqslant \frac{1}{4} |\langle \hat{S}_x \rangle|^2, \\ \langle \Delta\hat{S}_z^2 \rangle \langle \Delta\hat{S}_x^2 \rangle &\geqslant \frac{1}{4} |\langle \hat{S}_y \rangle|^2 \end{aligned} \tag{5}$$

spin squeezing will now be defined for the *spin operators* via conditions such as

$$\langle\Delta\hat{S}_x^2\rangle < \frac{1}{2}|\langle\hat{S}_z\rangle| \text{ and } \langle\Delta\hat{S}_y^2\rangle > \frac{1}{2}|\langle\hat{S}_z\rangle|,$$
$$\langle\Delta\hat{S}_y^2\rangle < \frac{1}{2}|\langle\hat{S}_x\rangle| \text{ and } \langle\Delta\hat{S}_z^2\rangle > \frac{1}{2}|\langle\hat{S}_x\rangle|,$$
$$\langle\Delta\hat{S}_z^2\rangle < \frac{1}{2}|\langle\hat{S}_y\rangle| \text{ and } \langle\Delta\hat{S}_x^2\rangle > \frac{1}{2}|\langle\hat{S}_y\rangle| \quad (6)$$

for $\hat{S}_x$ being squeezed compared to $\hat{S}_y$, and so on.

Note also that the Heisenberg uncertainty principle proof (based on $\langle(\Delta\hat{S}_\alpha + i\lambda\Delta\hat{S}_\beta)(\Delta\hat{S}_\alpha - i\lambda\Delta\hat{S}_\beta)\rangle \geqslant 0$ for all real λ) also establishes the general result for all quantum states

$$\langle\Delta\hat{S}_\alpha^2\rangle + \langle\Delta\hat{S}_\beta^2\rangle \geqslant |\langle\hat{S}_\gamma\rangle|, \quad (7)$$

where α, β and γ are x, y and z in cyclic order.

Other criteria for spin squeezing are also used, for example in the article by Wineland *et al* [17], where the focus is on spin squeezing for one component compared to *any* perpendicular component, and is set out in appendix B. A further special case is that of *planar squeezing* [18] where the Bloch vector is in one plane and the focus is on the spin fluctuation in this plane being squeezed compared to that perpendicular to the plane. This case is also discussed in appendix B.

Since the two new mode spin operators defined in equation (3) satisfy the standard angular momentum operator commutation rules, the usual Heisenberg Uncertainty rules analogous to (5) apply, so that spin squeezing can also exist in the two mode cases involving the *new spin operators* $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ as well. These uncertainty principle features also apply to multimode spin operators as defined in appendix B. The criteria for spin squeezing in the *multimode* case is of the same form as (6), but for completeness is included in appendix B.

It should be noted that finding spin squeezing for one principal spin operator $\hat{J}_y$ with respect to another $\hat{J}_x$ does *not* mean that there is spin squeezing for *any* of the old spin operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ In the case of the relative phase eigenstate (see section 3.7) $\hat{J}_y$ is squeezed with respect to $\hat{J}_x$—however none of the old spin components are spin squeezed.

2.4. Rotation operators and new modes

2.4.1. Rotation operators. The new spin operators are also related to the original spin operators via a *unitary rotation operator* $\hat{R}(\alpha, \beta, \gamma)$ parameterized in terms of Euler angles so that

$$\hat{J}_\xi = \hat{R}(\alpha, \beta, \gamma)\hat{S}_\xi\,\hat{R}(\alpha, \beta, \gamma)^{-1}, \quad (8)$$

where

$$\hat{R}(\alpha, \beta, \gamma) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma) \quad (9)$$

with $\hat{R}_\xi(\phi) = \exp(i\phi\hat{S}_\xi)$ describing a rotation about the ξ axis anticlockwise through an angle ϕ. Details for the rotation operators and matrices are set out in [7]. Note that equation (8) specifies a rotation of the vector spin operator rather than a rotation of the axes, so $\hat{J}_\xi$ ($\xi = x, y, z$) are the components of the rotated vector spin operator with respect to the original axes.

2.4.2. New mode operators. We can also see that the new spin operators are related to *new mode operators* $\hat{c}$ and $\hat{d}$ via

$$\hat{J}_x = (\hat{d}^\dagger\hat{c} + \hat{c}^\dagger\hat{d})/2, \qquad \hat{J}_y = (\hat{d}^\dagger\hat{c} - \hat{c}^\dagger\hat{d})/2i,$$
$$\hat{J}_z = (\hat{d}^\dagger\hat{d} - \hat{c}^\dagger\hat{c})/2, \quad (10)$$

where

$$\hat{c} = \hat{R}(\alpha, \beta, \gamma)\hat{a}\,\hat{R}(\alpha, \beta, \gamma)^{-1},$$
$$\hat{d} = \hat{R}(\alpha, \beta, \gamma)\hat{b}\,\hat{R}(\alpha, \beta, \gamma)^{-1}. \quad (11)$$

For the bosonic case a straight-forward calculation gives the new mode operators as

$$\hat{c} = \exp\left(\frac{1}{2}i\gamma\right)\left(\cos\left(\frac{\beta}{2}\right)\exp\left(\frac{1}{2}i\alpha\right)\hat{a} + \sin\left(\frac{\beta}{2}\right)\exp\left(-\frac{1}{2}i\alpha\right)\hat{b}\right),$$
$$\hat{d} = \exp\left(-\frac{1}{2}i\gamma\right)\left(-\sin\left(\frac{\beta}{2}\right)\exp\left(\frac{1}{2}i\alpha\right)\hat{a} + \cos\left(\frac{\beta}{2}\right)\exp\left(-\frac{1}{2}i\alpha\right)\hat{b}\right) \quad (12)$$

and it is easy to then check that $\hat{c}$ and $\hat{d}$ satisfy the expected non-zero bosonic commutation rules are $[\hat{e}, \hat{e}^\dagger] = \hat{1}$ ($\hat{e} = \hat{c}$ or $\hat{d}$) and that the *total boson number operator* is $\hat{N} = (\hat{d}^\dagger\hat{d} + \hat{c}^\dagger\hat{c})$. As $\hat{N}$ is invariant under unitary rotation operators it follows that $\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = (\hat{N}/2)(\hat{N}/2 + 1)$.

2.4.3. New modes. The new mode operators correspond to *new single particle states* $|\phi_c\rangle$, $|\phi_d\rangle$ where

$$|\phi_c\rangle = \exp\left(-\frac{1}{2}i\gamma\right)\left(\cos\left(\frac{\beta}{2}\right)\exp\left(-\frac{1}{2}i\alpha\right)|\phi_a\rangle + \sin\left(\frac{\beta}{2}\right)\exp\left(\frac{1}{2}i\alpha\right)|\phi_b\rangle\right),$$
$$|\phi_d\rangle = \exp\left(\frac{1}{2}i\gamma\right)\left(-\sin\left(\frac{\beta}{2}\right)\exp\left(-\frac{1}{2}i\alpha\right)|\phi_a\rangle + \cos\left(\frac{\beta}{2}\right)\exp\left(\frac{1}{2}i\alpha\right)|\phi_b\rangle\right) \quad (13)$$

These are two orthonormal quantum superpositions of the original single particle states $|\phi_a\rangle$, $|\phi_b\rangle$, and as such represent an *alternative choice* of modes that could be realized experimentally.

Equations (12) can be inverted to give the old mode operators via

$$\begin{aligned}\hat{a} &= \exp\left(-\frac{1}{2}\mathrm{i}\alpha\right)\left(\cos\left(\frac{\beta}{2}\right)\exp\left(-\frac{1}{2}\mathrm{i}\gamma\right)\hat{c}\right.\\ &\quad\left.-\sin\left(\frac{\beta}{2}\right)\exp\left(+\frac{1}{2}\mathrm{i}\gamma\right)\hat{d}\right)\\ \hat{b} &= \exp\left(+\frac{1}{2}\mathrm{i}\alpha\right)\left(\sin\left(\frac{\beta}{2}\right)\exp\left(\frac{1}{2}\mathrm{i}\gamma\right)\hat{c}\right.\\ &\quad\left.+\cos\left(\frac{\beta}{2}\right)\exp\left(-\frac{1}{2}\mathrm{i}\gamma\right)\hat{d}\right) \qquad (14)\end{aligned}$$

2.5. Old and new modes—coherence terms

The general non-entangled state for modes $\hat{a}$ and $\hat{b}$ is given by

$$\hat{\rho} = \sum_R P_R\, \hat{\rho}_R^A \otimes \hat{\rho}_R^B \qquad (15)$$

and as a consequence of the requirement that $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ are physical states for modes $\hat{a}$ and $\hat{b}$ satisfying the SSR, it follows that

$$\begin{aligned}\langle(\hat{a})^n\rangle_c &= \mathrm{Tr}(\hat{\rho}_R^A(\hat{a})^n) = 0,\\ \langle(\hat{a}^\dagger)^n\rangle_c &= \mathrm{Tr}(\hat{\rho}_R^A(\hat{a}^\dagger)^n) = 0,\\ \langle(\hat{b})^m\rangle_d &= \mathrm{Tr}(\hat{\rho}_R^B(\hat{b})^m) = 0,\\ \langle(\hat{b}^\dagger)^m\rangle_d &= \mathrm{Tr}(\hat{\rho}_R^B(\hat{b}^\dagger)^m) = 0. \qquad (16)\end{aligned}$$

Thus *coherence* terms are zero.

For our two-mode case we have also seen that the original choice of modes with annihilation operators $\hat{a}$ and $\hat{b}$ may be replaced by new modes with annihilation operators $\hat{c}$ and $\hat{d}$. Since the new modes are associated with new spin operators $\hat{J}_\xi$ ($\xi = x, y, z$) for which the covariance matrix is diagonal and where the diagonal elements give the variances, it is therefore also relevant to consider entanglement for the case where the sub-systems are modes $\hat{c}$ and $\hat{d}$, rather than $\hat{a}$ and $\hat{b}$. Consequently we also consider general non-entangled states for modes $\hat{c}$ and $\hat{d}$ in which the density operator is of the same form as (15), but with $\hat{\rho}_R^A \to \hat{\rho}_R^C$ and $\hat{\rho}_R^B \to \hat{\rho}_R^D$. Results analogous to (16) apply in this case, but with $\hat{a} \to \hat{c}$ and $\hat{b} \to \hat{d}$.

2.6. Quantum correlation functions and spin measurements

Finally, we note that the spin fluctuations can be related to *quantum correlation functions*. For example, it is easy to show that

$$\begin{aligned}\langle\Delta\hat{S}_x^2\rangle &= \frac{1}{4}(\langle(\hat{b}^\dagger)^2(\hat{a})^2\rangle + \langle(\hat{a}^\dagger)^2(\hat{b})^2\rangle\\ &\quad + 2\langle\hat{b}^\dagger\hat{a}^\dagger\hat{a}\hat{b}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\hat{a}\rangle)\\ &\quad - \frac{1}{4}(\langle(\hat{b}^\dagger\hat{a}\rangle^2 + \langle(\hat{a}^\dagger\hat{b}\rangle^2 + 2\langle(\hat{b}^\dagger\hat{a}\rangle\langle(\hat{a}^\dagger\hat{b}\rangle) \qquad (17)\end{aligned}$$

showing that $\langle\Delta\hat{S}_x^2\rangle$ is related to various first and second order quantum correlation functions. These can be measured experimentally and are given theoretically in terms of phase space integrals involving distribution functions to represent the density operator and phase space variables to represent the mode annihilation, creation operators.

3. Spin squeezing test for entanglement

With the general non-entangled state now required to be such that the density operators for the individual sub-systems must represent quantum states that conform to the SSR, the consequential link between entanglement in *two mode* bosonic systems and spin squeezing can now be established. We first consider spin squeezing for the original spin operators $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ and entangled states of the original modes $\hat{a}$, $\hat{b}$. and then for the principal spin operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ and entangled states of the related new modes $\hat{c}$, $\hat{d}$. We show [2] that spin squeezing in *any* spin component is a *sufficiency test* for entanglement of the two modes involved. Examples of entangled states that are not spin squeezed and states that are not entangled nor spin squeezed for one choice of mode sub-systems, but are entangled and spin squeezed for another choice are then presented.

Spin squeezing tests can also be established for *multi-mode* systems of identical massive bosons. For the common situation where there are two types of modes ($\hat{a}_i$ modes and $\hat{b}_i$ modes, such as when the bosons involve two different internal states), it turns out there are *three sub-cases* involved depending on different possible choices of the sub-systems. The bipartite *Case 1* involves two sub-systems, one consisting of all the $\hat{a}_i$ modes as sub-system *A* and the other consisting of all the $\hat{b}_i$ modes as sub-system *B*. *Case 2* involves $2n$ sub-systems, the Aith containing the mode $\hat{a}_i$ and the Bith containing the mode $\hat{b}_i$. *Case 3* involves n sub-systems, the ith containing the two modes $\hat{a}_i$ and $\hat{b}_i$. Spin squeezing as a test for entanglement applies for both Case 1 and Case 2, but only for Case 3 when for separable states each pair of modes involves a one particle state. A full discussion of the various cases and whether spin squeezing tests confirm entanglement is presented in appendix D.

3.1. Spin squeezing and entanglement—old modes

Firstly, the *mean* for a Hermitian operator $\hat{\Omega}$ in a mixed state

$$\hat{\rho} = \sum_R P_R\, \hat{\rho}_R \qquad (18)$$

is the *average* of means for separate components

$$\langle\hat{\Omega}\rangle = \sum_R P_R\, \langle\hat{\Omega}\rangle_R, \qquad (19)$$

where $\langle\hat{\Omega}\rangle_R = \mathrm{Tr}(\hat{\rho}_R\, \hat{\Omega})$.

Secondly, the *variance* for a Hermitian operator $\hat{\Omega}$ in a mixed state is always *never less* than the the *average* of the variances for the separate components (see [19])

$$\langle\Delta\hat{\Omega}^2\rangle \geqslant \sum_R P_R\, \langle\Delta\hat{\Omega}_R{}^2\rangle_R, \qquad (20)$$

where $\langle\Delta\widehat{\Omega}^2\rangle = \mathrm{Tr}(\widehat{\rho}\,\Delta\widehat{\Omega}^2)$ with $\Delta\widehat{\Omega} = \widehat{\Omega} - \langle\widehat{\Omega}\rangle$ and $\langle\Delta\widehat{\Omega}^2\rangle_R = \mathrm{Tr}(\widehat{\rho}_R\,\Delta\widehat{\Omega}_R^2)$ with $\Delta\widehat{\Omega}_R = \widehat{\Omega} - \langle\widehat{\Omega}\rangle_R$. To prove this result we have using (19) both for $\widehat{\Omega}$ and $\widehat{\Omega}^2$

$$\begin{aligned}\langle\Delta\widehat{\Omega}^2\rangle &= \langle\widehat{\Omega}^2\rangle - \langle\widehat{\Omega}\rangle^2\\ &= \sum_R P_R(\langle\widehat{\Omega}^2\rangle_R - \langle\widehat{\Omega}\rangle_R^2) + \sum_R P_R\langle\widehat{\Omega}\rangle_R^2\\ &\quad - \left(\sum_R P_R\langle\widehat{\Omega}\rangle_R\right)^2\\ &= \sum_R P_R\langle\Delta\widehat{\Omega}_R^2\rangle_R + \sum_R P_R\langle\widehat{\Omega}\rangle_R^2\\ &\quad - \left(\sum_R P_R\,|\langle\widehat{\Omega}\rangle_R|\right)^2.\end{aligned} \tag{21}$$

The variance result (20) follows because the sum of the last two terms is always $\geqslant 0$ using the result (129) in appendix E of paper 1, with $C_R = \langle\widehat{\Omega}\rangle_R^2$, $\sqrt{C_R} = |\langle\widehat{\Omega}\rangle_R|$—which are real and positive.

3.1.1. Mean values for $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$. Next, we find the *mean values* of the spin operators for the product state $\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$

$$\begin{aligned}\langle\widehat{S}_x\rangle_R &= \frac{1}{2}(\langle\widehat{b}^\dagger\rangle_R\langle\widehat{a}\rangle_R + \langle\widehat{a}^\dagger\rangle_R\langle\widehat{b}\rangle_R) = 0,\\ \langle\widehat{S}_y\rangle_R &= \frac{1}{2i}(\langle\widehat{b}^\dagger\rangle_R\langle\widehat{a}\rangle_R - \langle\widehat{a}^\dagger\rangle_R\langle\widehat{b}\rangle_R) = 0\end{aligned} \tag{22}$$

and

$$\langle\widehat{S}_z\rangle_R = \frac{1}{2}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R - \langle\widehat{a}^\dagger\widehat{a}\rangle_R) \tag{23}$$

for SSR compliant sub-system states using (16), and thus using (19) the *overall mean* values for the *separable* state is

$$\langle\widehat{S}_x\rangle = 0, \qquad \langle\widehat{S}_y\rangle = 0 \tag{24}$$

and

$$\langle\widehat{S}_z\rangle = \sum_R P_R\frac{1}{2}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R - \langle\widehat{a}^\dagger\widehat{a}\rangle_R)). \tag{25}$$

Hence if either $\langle\widehat{S}_x\rangle \neq 0$ or $\langle\widehat{S}_y\rangle \neq 0$ the state must be entangled. This may be called the *Bloch vector* test. This result will also have significance later.

3.1.2. Variances for $\widehat{S}_x$ and $\widehat{S}_y$. Next we calculate $\langle\Delta\widehat{S}_x^2\rangle_R$, $\langle\Delta\widehat{S}_y^2\rangle_R$ and $\langle\widehat{S}_x\rangle_R$, $\langle\widehat{S}_y\rangle_R$, $\langle\widehat{S}_z\rangle_R$ for the case of the *separable state* (15) where $\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$. From equation (1) we find that

$$\begin{aligned}\widehat{S}_x^2 &= \frac{1}{4}((\widehat{b}^\dagger)^2(\widehat{a})^2 + \widehat{b}^\dagger\widehat{b}\widehat{a}\widehat{a}^\dagger + \widehat{a}^\dagger\widehat{a}\widehat{b}\widehat{b}^\dagger + (\widehat{b})^2(\widehat{a}^\dagger)^2),\\ \widehat{S}_y^2 &= -\frac{1}{4}((\widehat{b}^\dagger)^2(\widehat{a})^2 - \widehat{b}^\dagger\widehat{b}\widehat{a}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a}\widehat{b}\widehat{b}^\dagger + (\widehat{b})^2(\widehat{a}^\dagger)^2)\end{aligned} \tag{26}$$

so that on taking the trace with $\widehat{\rho}_R$ and using equations (16) we get after applying the commutation rules $[\widehat{e}, \widehat{e}^\dagger] = \widehat{1}$ ($\widehat{e} = \widehat{a}$ or $\widehat{b}$)

$$\begin{aligned}\langle\widehat{S}_x^2\rangle_R &= \frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R),\\ \langle\widehat{S}_y^2\rangle_R &= \frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R).\end{aligned} \tag{27}$$

Hence using (22) for $\langle\widehat{S}_x\rangle_R$ and $\langle\widehat{S}_y\rangle_R$ we see finally that the variances are

$$\begin{aligned}\langle\Delta\widehat{S}_x^2\rangle_R &= \frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R),\\ \langle\Delta\widehat{S}_y^2\rangle_R &= \frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R)\end{aligned} \tag{28}$$

and therefore from equation (20)

$$\begin{aligned}\langle\Delta\widehat{S}_x^2\rangle &\geqslant \sum_R P_R\frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R)\\ &\quad + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R),\\ \langle\Delta\widehat{S}_y^2\rangle &\geqslant \sum_R P_R\frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R)\\ &\quad + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R).\end{aligned} \tag{29}$$

Now using (25) for $\langle\widehat{S}_z\rangle$ we see that

$$\begin{aligned}\frac{1}{2}|\langle\widehat{S}_z\rangle| &\leqslant \sum_R P_R\frac{1}{4}|(\langle\widehat{b}^\dagger\widehat{b}\rangle_R - \langle\widehat{a}^\dagger\widehat{a}\rangle_R))|\\ &\leqslant \sum_R P_R\frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R))\end{aligned} \tag{30}$$

and thus for any non-entangled state of modes $\widehat{a}$ and $\widehat{b}$

$$\begin{aligned}&\langle\Delta\widehat{S}_x^2\rangle - \frac{1}{2}|\langle\widehat{S}_z\rangle|\\ &\geqslant \sum_R P_R\frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R)\\ &\quad - \sum_R P_R\frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R))\\ &\geqslant \sum_R P_R\frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R)\\ &\geqslant 0.\end{aligned} \tag{31}$$

Similar final steps show that $\langle\Delta\widehat{S}_y^2\rangle - \frac{1}{2}|\langle\widehat{S}_z\rangle| \geqslant 0$ for all non-entangled state of modes $\widehat{a}$ and $\widehat{b}$.

This shows that for the general non-entangled state with modes $\widehat{a}$ and $\widehat{b}$ as the sub-systems, the variances for two of the principal spin fluctuations $\langle\Delta\widehat{S}_x^2\rangle$ and $\langle\Delta\widehat{S}_y^2\rangle$ are *both* greater than $\frac{1}{2}|\langle\widehat{S}_z\rangle|$, and hence there is no spin squeezing for $\widehat{S}_x$ compared to $\widehat{S}_y$ (or vice versa). Note that as $|\langle\widehat{S}_y\rangle| = 0$, the quantity $\sqrt{(|\langle\widehat{S}_{\perp 1}\rangle|^2 + |\langle\widehat{S}_{\perp 2}\rangle|^2)}$ is the same as $|\langle\widehat{S}_z\rangle|$, so the alternative criterion in appendix B equation (201) is the same as that in equation (6) which is used here.

It is easy to see from (24) that

$$\langle\Delta\widehat{S}_x^2\rangle - \frac{1}{2}|\langle\widehat{S}_y\rangle| \geqslant 0, \qquad \langle\Delta\widehat{S}_y^2\rangle - \frac{1}{2}|\langle\widehat{S}_x\rangle| \geqslant 0 \qquad (32)$$

for any non-entangled state of modes $\widehat{a}$ and $\widehat{b}$. This completes the set of inequalities for the variances of $\widehat{S}_x$ and $\widehat{S}_y$. These last inequalities are of course trivially true and amount to no more than showing that the variances $\langle\Delta\widehat{S}_x^2\rangle$ and $\langle\Delta\widehat{S}_y^2\rangle$ are not negative.

3.1.3. Variance for $\widehat{S}_z$. For the other principal spin fluctuation we find that for separable states

$$\langle\Delta\widehat{S}_z^2\rangle_R = \frac{1}{4}(\langle(\widehat{b}^\dagger\widehat{b} - \langle\widehat{b}^\dagger\widehat{b}\rangle_R)(\widehat{b}^\dagger\widehat{b} - \langle\widehat{b}^\dagger\widehat{b}\rangle_R)\rangle_R + \langle(\widehat{a}^\dagger\widehat{a} - \langle\widehat{a}^\dagger\widehat{a}\rangle_R)(\widehat{a}^\dagger\widehat{a} - \langle\widehat{a}^\dagger\widehat{a}\rangle_R)\rangle_R \qquad (33)$$

so that using (20)

$$\langle\Delta\widehat{S}_z^2\rangle \geqslant \sum_R P_R\,\frac{1}{4}(\langle(\widehat{b}^\dagger\widehat{b} - \langle\widehat{b}^\dagger\widehat{b}\rangle_R)^2\rangle_R + \langle(\widehat{a}^\dagger\widehat{a} - \langle\widehat{a}^\dagger\widehat{a}\rangle_R)^2\rangle_R. \qquad (34)$$

From equation (24) it follows that

$$\langle\Delta\widehat{S}_z^2\rangle - \frac{1}{2}|\langle\widehat{S}_x\rangle| \geqslant \sum_R P_R\,\frac{1}{4}(\langle(\widehat{b}^\dagger\widehat{b} - \langle\widehat{b}^\dagger\widehat{b}\rangle_R)^2\rangle_R + \langle(\widehat{a}^\dagger\widehat{a} - \langle\widehat{a}^\dagger\widehat{a}\rangle_R)^2\rangle_R \geqslant 0. \qquad (35)$$

Similarly $\langle\Delta\widehat{S}_z^2\rangle - \frac{1}{2}|\langle\widehat{S}_y\rangle| \geqslant 0$. Again, these results are trivial and just show that the variances are non-negative.

3.1.4. No spin squeezing for separable states. So overall, we have for the general non-entangled state of modes $\widehat{a}$ and $\widehat{b}$

$$\begin{aligned}
&\langle\Delta\widehat{S}_x^2\rangle \geqslant \frac{1}{2}|\langle\widehat{S}_z\rangle| \text{ and } \langle\Delta\widehat{S}_y^2\rangle \geqslant \frac{1}{2}|\langle\widehat{S}_z\rangle|,\\
&\langle\Delta\widehat{S}_y^2\rangle \geqslant \frac{1}{2}|\langle\widehat{S}_x\rangle| \text{ and } \langle\Delta\widehat{S}_z^2\rangle \geqslant \frac{1}{2}|\langle\widehat{S}_x\rangle|,\\
&\langle\Delta\widehat{S}_z^2\rangle \geqslant \frac{1}{2}|\langle\widehat{S}_y\rangle| \text{ and } \langle\Delta\widehat{S}_x^2\rangle \geqslant \frac{1}{2}|\langle\widehat{S}_y\rangle|. \qquad (36)
\end{aligned}$$

The first result tells us that for *any* non-entangled state of modes $\widehat{a}$ and $\widehat{b}$ the spin operator $\widehat{S}_x$ is *not* squeezed compared to $\widehat{S}_y$ (or vice-versa). The same is also true for the other pairs of spin operators, as we will now see.

3.1.5. Spin squeezing tests for entanglement. The key value of these results is the *spin squeezing test* for *entanglement*. We see that from the first inequality in (36) for separable states, that *if* for a quantum state we find that

$$\langle\Delta\widehat{S}_x^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle| \quad \text{or} \quad \langle\Delta\widehat{S}_y^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle| \qquad (37)$$

then the state *must* be entangled for modes $\widehat{a}$ and $\widehat{b}$. Thus we only need to have spin squeezing in *either* of $\widehat{S}_x$ with respect to $\widehat{S}_y$ *or* vice-versa to demonstrate entanglement. Note that one cannot have *both* $\langle\Delta\widehat{S}_x^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle|$ and $\langle\Delta\widehat{S}_y^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle|$ etc due to the Heisenberg uncertainty principle.

Because $\langle\widehat{S}_x\rangle_\rho = \langle\widehat{S}_y\rangle_\rho = 0$ the second and third results in (36) merely show that $\langle\Delta\widehat{S}_x^2\rangle \geqslant 0$, $\langle\Delta\widehat{S}_y^2\rangle \geqslant 0$ and $\langle\Delta\widehat{S}_z^2\rangle \geqslant 0$ for SSR compliant non-entangled states, it may be thought that no conclusion follows regarding the spin squeezing involving $\widehat{S}_z$ for entangled states. This is not the case. *If* for a given state we find that

$$\langle\Delta\widehat{S}_y^2\rangle < \frac{1}{2}|\langle\widehat{S}_x\rangle| \quad \text{or} \quad \langle\Delta\widehat{S}_z^2\rangle < \frac{1}{2}|\langle\widehat{S}_x\rangle| \qquad (38)$$

or

$$\langle\Delta\widehat{S}_z^2\rangle < \frac{1}{2}|\langle\widehat{S}_y\rangle| \quad \text{or} \quad \langle\Delta\widehat{S}_x^2\rangle < \frac{1}{2}|\langle\widehat{S}_y\rangle| \qquad (39)$$

then the state *must* be entangled. This is because if any of these situations apply then *at least one* of $\langle\widehat{S}_x\rangle_\rho$ or $\langle\widehat{S}_y\rangle_\rho$ must be non-zero. But as we have seen from (24) both of the quantities are zero in non-entangled states. Thus we only need to have spin squeezing in *either* of $\widehat{S}_z$ with respect to $\widehat{S}_y$ *or* vice-versa or spin squeezing in *either* of $\widehat{S}_z$ with respect to $\widehat{S}_x$ *or* vice-versa to demonstrate entanglement.

Hence the general conclusion stated in [2], that spin squeezing in *any* spin operator $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ shows that the state must be entangled for modes $\widehat{a}$ and $\widehat{b}$. The presence of spin squeezing is a conclusive test for entanglement. Note that the reverse is not true—there are many entangled states that are *not* spin squeezed. A notable example is the particular *binomial* state $|\Phi\rangle = ((\widehat{a} + \widehat{b})^\dagger/\sqrt{2})^N/\sqrt{N!}\,|0\rangle$ for which $\langle\widehat{S}_x\rangle_\rho = N/2$, $\langle\widehat{S}_y\rangle_\rho = \langle\widehat{S}_z\rangle_\rho = 0$ and $\langle\Delta\widehat{S}_y^2\rangle_\rho = \langle\Delta\widehat{S}_z^2\rangle_\rho = N/4$, $\langle\Delta\widehat{S}_x^2\rangle_\rho = 0$ (see [7]). The spin fluctuations in $\widehat{S}_y$ and $\widehat{S}_z$ correspond to the *standard quantum limit*.

This is a *key result* for two mode entanglement. *All* spin squeezed states are *entangled*. We emphasize again that the converse is not true. *Not all* entangled two mode states are *spin squeezed*. This important distinction is not always recognized—entanglement and spin squeezing are two *distinct* features of a two mode quantum state that do not always occur together.

For the two orthogonal spin operator components (199) (appendix B) in the *xy* plane $\widehat{S}_{\perp 1}$ and $\widehat{S}_{\perp 2}$ it is then straightforward to show that

$$\text{If} \quad \langle\Delta\widehat{S}_{\perp 1}^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle| \qquad (40)$$

or

$$\text{If} \quad \langle\Delta\widehat{S}_{\perp 2}^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle| \qquad (41)$$

that is, if $\widehat{S}_{\perp 1}$ is squeezed compared to $\widehat{S}_{\perp 2}$ or vice versa—then the state must be entangled. Spin squeezing in *any* of the spin operator component in the *xy* plane will demonstrate entanglement.

The derivation of the spin squeezing test for this two mode system of identical bosons was based on the requirements that the quantum state complied with both the symmetrization principle and the global superselection rule for total boson numbers, together with the subsystem states complying with the local particle number superselection rule in the case of. separable states. However, as discussed at length in paper I, there are some papers such as [4] in which the local particle number superselection rule is not applied to separable states—leading to the concept of 'separable but non-local' states (which are regarded here as entangled, not separable). Accordingly, such other approaches would result in different tests for what they define as entanglement. It is therefore of some interest to consider what interpretation could be placed on observing spin squeezing in any spin component, both from the point of view about entanglement presented here and from that in other papers such as [4]. This discussion is set out in appendix C. Figures 2 and 3 (within the supplementary material) therein depict the various types of quantum states involved—separable, 'separable but non-local', other entangled states, as well as indicating whch are spin squeezed.

3.1.6. Inequality for $|\langle \widehat{S}_Z\rangle|$. Of the results for a *non-entangled* physical state for modes $\widehat{a}$ and $\widehat{b}$ we will later find it particularly important to consider the first of (36)

$$\langle \Delta\widehat{S}_x^2\rangle \geqslant \frac{1}{2}|\langle \widehat{S}_z\rangle| \quad \text{and} \quad \langle \Delta\widehat{S}_y^2\rangle \geqslant \frac{1}{2}|\langle \widehat{S}_z\rangle|. \tag{42}$$

This is because we can show that for any quantum state

$$|\langle \widehat{S}_z\rangle| = \left| \left\langle \frac{1}{2}(\widehat{n}_b - \widehat{n}_a)\right\rangle \right| \leqslant \frac{1}{2}(|\langle \widehat{n}_b\rangle| + |\langle \widehat{n}_a\rangle|) = \frac{1}{2}\langle \widehat{N}\rangle \tag{43}$$

there is an inequality involving $|\langle \widehat{S}_z\rangle|$ and the mean number of bosons $\langle \widehat{N}\rangle$ in the two mode system. Note that there *are* some entangled states (see section 3.5) for which $\langle \Delta\widehat{S}_x^2\rangle$ and $\langle \Delta\widehat{S}_y^2\rangle$ are both greater than $\frac{1}{2}|\langle \widehat{S}_z\rangle|$, since all that has been proven is that for *all* non-entangled states we must have *both* $\langle \Delta\widehat{S}_x^2\rangle \geqslant \frac{1}{2}|\langle \widehat{S}_z\rangle|$ and $\langle \Delta\widehat{S}_y^2\rangle \geqslant \frac{1}{2}|\langle \widehat{S}_z\rangle|$.

Hence we may conclude that spin squeezing in any of the original spin fluctuations $\widehat{S}_x$, $\widehat{S}_y$ or $\widehat{S}_z$ requires the quantum state to be entangled for the modes $\widehat{a}$ and $\widehat{b}$ as the sub-systems. Similarly, we may conclude that spin squeezing in any of the principal spin fluctuations $\widehat{J}_x$, $\widehat{J}_y$ or $\widehat{J}_z$ requires the quantum state to be entangled for the modes $\widehat{c}$ and $\widehat{d}$ as the sub-systems, these modes being associated with the principal spin fluctuations via equation (10). Although finding spin squeezing tells us that the state is entangled, there are however no simple relationships between the measures of entanglement and those of spin squeezing, so the linkage is essentially a qualitative one. For general quantum states, measures of entanglement for the specific situation of two sub-systems (bipartite entanglement) are reviewed in [20].

3.2. Spin squeezing and entanglement—new modes

It is also of some interest to consider spin squeezing for the new spin operators $\widehat{J}_x$, $\widehat{J}_y$, $\widehat{J}_z$ with the new modes $\widehat{c}$ and $\widehat{d}$ as the sub-systems, where these spin operators are associated with a diagonal covariance matrix. The definition of spin squeezing in this case is set out analogous to that in appendix B equation (204). In this case the general non-entangled state for the *new* modes is given by

$$\widehat{\rho} = \sum_R P_R\, \widehat{\rho}_R^C \otimes \widehat{\rho}_R^D \tag{44}$$

with the $\widehat{\rho}_R^C$ and $\widehat{\rho}_R^D$ representing physical states for modes $\widehat{c}$ and $\widehat{d}$, and where results analogous to equations (16) apply. The same treatment applies as for spin operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ with the modes $\widehat{a}$ and $\widehat{b}$ as the sub-systems and leads to the result for a *non-entangled* state of modes $\widehat{c}$ and $\widehat{d}$

$$\langle \Delta\widehat{J}_x^2\rangle \geqslant \frac{1}{2}|\langle \widehat{J}_z\rangle| \quad \text{and} \quad \langle \Delta\widehat{J}_y^2\rangle \geqslant \frac{1}{2}|\langle \widehat{J}_z\rangle \tag{45}$$

showing that neither $\widehat{J}_x$ or $\widehat{J}_y$ is spin squeezed for the general non-entangled state for modes $\widehat{c}$ and $\widehat{d}$ given in equation (12). We also have

$$\langle \widehat{J}_x\rangle = \sum_R P_R\,\langle \widehat{J}_x\rangle_R = 0 \qquad \langle \widehat{J}_y\rangle = \sum_R P_R\,\langle \widehat{J}_y\rangle_R = 0 \tag{46}$$

so all the results analogous to equation (36) also follow. Following similar arguments as in section 3.1 we may also conclude that spin squeezing in *any* of the original spin fluctuations requires the quantum state to be entangled when the original modes $\widehat{c}$ and $\widehat{d}$ are the sub-systems. Thus the *entanglement test* is

$$\text{If} \quad \langle \Delta\widehat{J}_x^2\rangle < \frac{1}{2}|\langle \widehat{J}_z\rangle| \quad \text{or} \quad \langle \Delta\widehat{J}_y^2\rangle < \frac{1}{2}|\langle \widehat{J}_z\rangle| \tag{47}$$

or

$$\text{If} \quad \langle \Delta\widehat{J}_y^2\rangle < \frac{1}{2}|\langle \widehat{J}_x\rangle| \quad \text{or} \quad \langle \Delta\widehat{J}_z^2\rangle < \frac{1}{2}|\langle \widehat{J}_x\rangle| \tag{48}$$

or

$$\text{If} \quad \langle \Delta\widehat{J}_z^2\rangle < \frac{1}{2}|\langle \widehat{J}_y\rangle| \quad \text{or} \quad \langle \Delta\widehat{J}_x^2\rangle < \frac{1}{2}|\langle \widehat{J}_y\rangle| \tag{49}$$

then we have an entangled state for the original modes $\widehat{c}$ and $\widehat{d}$.

The result (46) also provides a *Bloch vector* entanglement test—if either $\langle \widehat{J}_x\rangle \neq 0$ or $\langle \widehat{J}_y\rangle \neq 0$ the state must be entangled.

Hence we have seen that spin squeezing—either of the new or original spin operators requires entanglement of the new or original modes. Which spin operators to consider depends on which pairs of modes are being tested for entanglement.

3.3. Bloch vector entanglement test

We have seen for the general non-entangled states of modes $\hat{c}$ and $\hat{d}$ or of modes $\hat{a}$ and $\hat{b}$ that

$$\langle\hat{J}_x\rangle = 0, \qquad \langle\hat{J}_y\rangle = 0, \tag{50}$$

$$\langle\hat{S}_x\rangle = 0, \qquad \langle\hat{S}_y\rangle = 0. \tag{51}$$

Hence the two mode *Bloch vector entanglement* tests

$$\begin{aligned} \langle\hat{J}_x\rangle \neq 0 \quad &\text{or} \quad \langle\hat{J}_y\rangle \neq 0 \\ \langle\hat{S}_x\rangle \neq 0 \quad &\text{or} \quad \langle\hat{S}_y\rangle \neq 0 \end{aligned} \tag{52}$$

for modes $\hat{c}$ and $\hat{d}$ or of modes $\hat{a}$ and $\hat{b}$. As shown in appendix D, for the *multi-mode* case the same Bloch vector test applies for Case 1, where there are just two sub-systems each consisting of all the modes $\hat{a}_i$ or all the modes $\hat{b}_i$ and in Case 2, where there are $2n$ sub-systems consisting of all the modes $\hat{a}_i$ and all the modes $\hat{b}_i$.

From equations (10) and (1) these results are equivalent to

$$\langle\hat{d}\,\hat{c}^{\dagger}\rangle = 0, \qquad \langle\hat{c}\,\hat{d}^{\dagger}\rangle = 0, \tag{53}$$

$$\langle\hat{b}\,\hat{a}^{\dagger}\rangle = 0, \qquad \langle\hat{a}\,\hat{b}^{\dagger}\rangle = 0. \tag{54}$$

Hence we find further *tests* for *entangled states* of modes $\hat{c}$ and $\hat{d}$ or of modes $\hat{a}$ and $\hat{b}$

$$|\langle\hat{d}\,\hat{c}^{\dagger}\rangle|^2 > 0, \qquad |\langle\hat{c}\,\hat{d}^{\dagger}\rangle|^2 > 0, \tag{55}$$

$$|\langle\hat{b}\,\hat{a}^{\dagger}\rangle|^2 > 0, \qquad |\langle\hat{a}\,\hat{b}^{\dagger}\rangle|^2 > 0. \tag{56}$$

As we will see in section 4, these tests are particular cases with $m = n = 1$ of the simpler entanglement test in equation (121) that applies for the situation in the present paper where non-entangled states are required to satisfy the SSR.

3.4. Entanglement test for number difference and sum

There is also a further spin squeezing test involving the operator $\hat{S}_z$, which is equal to half the *number difference* $\frac{1}{2}(\hat{n}_b - \hat{n}_a)$. We note that simultaneous eigenstates of $\hat{n}_a$ and $\hat{n}_b$ exist, which are also eigenstates of the total number operator. For such states the variances $\langle\Delta\hat{n}_a^2\rangle$, $\langle\Delta\hat{n}_b^2\rangle$, $\langle\Delta\hat{S}_z^2\rangle$ and $\langle\Delta\hat{N}^2\rangle$ are all zero, which does not suggest that useful general inequalities for these variances would be found. However, a useful entanglement test—which does not require SSR compliance can be found. For the variance of $\hat{S}_z$ in a separable state we have

$$\begin{aligned} \langle\Delta\hat{S}_z^2\rangle &\geqslant \sum_R P_R\langle\Delta\hat{S}_z^2\rangle_R = \sum_R P_R(\langle\hat{S}_z^2\rangle_R - \langle\hat{S}_z\rangle_R^2) \\ &= \frac{1}{4}\sum_R P_R(\langle\hat{n}_b^2\rangle_R + \langle\hat{n}_a^2\rangle_R - 2\langle\hat{n}_b\rangle_R\langle\hat{n}_a\rangle_R \\ &\quad - \langle\hat{n}_b\rangle_R^2 - \langle\hat{n}_a\rangle_R^2 + 2\langle\hat{n}_b\rangle_R\langle\hat{n}_a\rangle_R) \\ &= \frac{1}{4}\sum_R P_R(\langle\Delta\hat{n}_b^2\rangle_R + \langle\Delta\hat{n}_a^2\rangle_R). \end{aligned} \tag{57}$$

For such a separable state we also find

$$\langle\Delta\hat{N}^2\rangle \geqslant \sum_R P_R(\langle\Delta\hat{n}_b^2\rangle_R + \langle\Delta\hat{n}_a^2\rangle_R). \tag{58}$$

This leads to the useful if somewhat qualitative test that if we have a state with a *large* fluctuation in the total boson number and a *small* fluctuation in the number difference, then it is likely to be an entangled state. If it was separable and $\langle\Delta\hat{N}^2\rangle$ is large, then $\langle\Delta\hat{S}_z^2\rangle$ is likely to be large. There is also the converse test —if we have a state with a small fluctuation in the total boson number and a large fluctuation in the number difference, then it is likely to be an entangled state.

3.5. Entangled states that are non spin-squeezed—NOON state

One such example is the generalized N boson *NOON state* defined as

$$\begin{aligned} \hat{\rho} &= |\Phi\rangle\langle\Phi|, \\ |\Phi\rangle &= \cos\theta\,\frac{(\hat{a}^{\dagger})^N}{\sqrt{N!}}|0\rangle + \sin\theta\,\frac{(\hat{b}^{\dagger})^N}{\sqrt{N!}}|0\rangle \\ &= \cos\theta\left|\frac{N}{2}, -\frac{N}{2}\right\rangle + \sin\theta\left|\frac{N}{2}, +\frac{N}{2}\right\rangle \end{aligned} \tag{59}$$

which is an entangled state for modes $\hat{a}$ and $\hat{b}$ in all cases except where $\cos\theta$ or $\sin\theta$ is zero. In the last form the state is expressed in terms of the eigenstates for $(\underset{\rightarrow}{\hat{S}})^2$ and $\hat{S}_z$, as detailed in [7].

A straight-forward calculation gives

$$\begin{aligned} &\langle\hat{S}_x\rangle = 0, \qquad \langle\hat{S}_y\rangle = 0, \qquad \langle\hat{S}_z\rangle = -\frac{N}{2}\cos 2\theta, \\ &\langle\Delta\hat{S}_x^2\rangle = \frac{N}{4}, \qquad \langle\Delta\hat{S}_y^2\rangle = \frac{N}{4}, \\ &\langle\Delta\hat{S}_z^2\rangle = \frac{N^2}{4}(1 - \cos^2 2\theta) \end{aligned} \tag{60}$$

for $N > 1$, so that using the criteria for spin squeezing given in equation (6) we see that as $\langle\Delta\hat{S}_x^2\rangle - \frac{1}{2}|\langle\hat{S}_z\rangle| \geqslant 0$, etc, and hence spin squeezing does not occur for this entangled state.

3.6. Non-entangled states that are non spin squeezed—binomial state

Of course from the previous section *any* non entangled state is definitely not spin squeezed. A specific example illustrating this is the N boson binomial state given by

$$\begin{aligned} \hat{\rho} &= |\Phi\rangle\langle\Phi|, \\ |\Phi\rangle &= \frac{(-\hat{c}^{\dagger})^N}{\sqrt{N!}}|0\rangle, \end{aligned} \tag{61}$$

where $\hat{c}$ and $\hat{d}$ are given by equation (12) with Euler angles $\alpha = -\pi + \chi$, $\beta = -2\theta$ and $\gamma = -\pi$, we find that

$$\begin{aligned} \hat{c} &= -\cos\theta\exp\left(\frac{1}{2}\mathrm{i}\chi\right)\hat{a} - \sin\theta\exp\left(-\frac{1}{2}\mathrm{i}\chi\right)\hat{b} = -\hat{a}_1 \\ \hat{d} &= \sin\theta\exp\left(\frac{1}{2}\mathrm{i}\chi\right)\hat{a} - \cos\theta\exp\left(-\frac{1}{2}\mathrm{i}\chi\right)\hat{b} = -\hat{a}_2 \end{aligned} \tag{62}$$

where the mode operators $\hat{a}_1$ and $\hat{a}_2$ are as defined in [7] (see equation (53) therein). The new spin angular momentum operators $\widehat{J}_\xi$ ($\xi = x, y, z$) are the same as those defined in [7] (see equations (64) therein) and in [7] it has been shown (see equation (60) therein) for the same binomial state as in (61) that

$$\langle \widehat{J}_x \rangle = 0, \qquad \langle \widehat{J}_y \rangle = 0, \qquad \langle \widehat{J}_z \rangle = -\frac{N}{2},$$
$$\langle \Delta \widehat{J}_x^2 \rangle = \frac{N}{4}, \qquad \langle \Delta \widehat{J}_y^2 \rangle = \frac{N}{4}, \qquad \langle \Delta \widehat{J}_z^2 \rangle = 0 \tag{63}$$

(see equations (162) and (176) therein). Hence the binomial state is not spin squeezed since $\langle \Delta \widehat{J}_x^2 \rangle = \langle \Delta \widehat{J}_y^2 \rangle = \frac{1}{2} |\langle \widehat{J}_z \rangle|$. It is of course a *minimum uncertainty state* with spin fluctuations at the *standard quantum limit*. Here $\sqrt{\langle \Delta \widehat{J}_{x,y}^2 \rangle} / |\langle \widehat{J}_z \rangle| = 1/\sqrt{N}$. Clearly, it is a non-entangled state for modes $\widehat{c}$ and $\widehat{d}$, being the product of a number state for mode $\widehat{c}$ with the vacuum state for mode $\widehat{d}$.

Note that from the point of view of the original modes $\widehat{a}$ and $\widehat{b}$, this is an entangled state. so the question is: Is it a spin squeezed state with respect to the original spin operators $\widehat{S}_\xi$ ($\xi = x, y, z$)? The Bloch vector and variances for this binomial state are given in [7] (see equation (163) in the main paper and equation (410) in the appendix). The results include:

$$\langle \widehat{S}_z \rangle = -\frac{N}{2}\cos 2\theta,$$
$$\langle \Delta \widehat{S}_x^2 \rangle = \frac{N}{4}(\cos^2 2\theta \cos^2 \chi + \sin^2 \chi),$$
$$\langle \Delta \widehat{S}_y^2 \rangle = \frac{N}{4}(\cos^2 2\theta \sin^2 \chi + \cos^2 \chi). \tag{64}$$

This gives $\langle \Delta \widehat{S}_x^2 \rangle \langle \Delta \widehat{S}_y^2 \rangle - \frac{1}{4} |\langle \widehat{S}_z \rangle|^2 = \frac{1}{16} N^2 (\cos^2 2\theta - 1)^2 \cos^2 \chi \sin^2 \chi \geqslant 0$ as required for the Heisenberg uncertainty principle. With $\chi = 0$ we have $\langle \Delta \widehat{S}_x^2 \rangle = \frac{N}{4}\cos^2 2\theta$ and $\langle \Delta \widehat{S}_y^2 \rangle = \frac{N}{4}$, while $\frac{1}{2}|\langle \widehat{S}_z \rangle| = \frac{N}{4}|\cos 2\theta|$. As $\langle \Delta \widehat{S}_x^2 \rangle < \frac{1}{2}|\langle \widehat{S}_z \rangle|$ there is spin squeezing in $\widehat{S}_x$ for this entangled state of modes $\widehat{a}$ and $\widehat{b}$, though not of course for the new spin operator $\widehat{J}_x$ since this state is non-entangled for modes $\widehat{c}$ and $\widehat{d}$. This example illustrates the need to carefully define spin squeezing and entanglement in terms of related sets of spin operators and modes. The same state is entangled with respect to one choice of modes—and spin squeezing occurs, while it is non-entangled with respect to another set of modes—and no spin squeezing occurs.

To summarize—with a physically based definition of non-entangled states for bosonic systems with two modes (related to the principal spin operators that have a diagonal covariance matrix) being the sub-systems and with a criterion for spin squeezing that focuses on these principal spin fluctuations, it seen that while non-entangled states are never spin squeezed and therefore although entanglement is a necessary condition for spin squeezing, it is not a sufficient one. There are entangled states that are not spin squeezed. Furthermore, as there is no simple quantitative links between measures of spin squeezing and those for entanglement, the mere presence of spin squeezing only demonstrates the qualitative result that the quantum state is entangled. Nevertheless, for high precision measurements based on spin operators where the primary emphasis is on how much spin squeezing can be achieved, knowing that entangled states are needed provides an impetus for studying such states and how they might be produced.

3.7. Entangled states that are spin squeezed—relative phase eigenstate

As an example of an entangled state that is spin squeezed we consider the relative phase eigenstate $\left| \frac{N}{2}, \theta_p \right\rangle$ for a two mode system in which there are N bosons. For modes with annihilation operators $\widehat{a}, \widehat{b}$ the *relative phase eigenstate* is defined as

$$\left| \frac{N}{2}, \theta_p \right\rangle = \frac{1}{\sqrt{N+1}} \sum_{k=-N/2}^{N/2} \exp(ik\theta_p) \frac{(\widehat{a}^\dagger)^{N/2-k}}{\sqrt{(N/2-k)!}} \times \frac{(\widehat{b}^\dagger)^{N/2+k}}{\sqrt{(N/2+k)!}} |0\rangle, \tag{65}$$

where the relative phase $\theta_p = p(2\pi/(N+1))$ with $p = -N/2, -N/2+1, \ldots, +N/2$, is an eigenvalue of the relative phase Hermitian operator of the type introduced by Barnett and Pegg [21] (see [7] and references therein). Note that the eigenvalues form a quasi-continuum over the range $-\pi$ to $+\pi$, with a small separation between neighboring phases of $O(1/N)$. The relative phase state is consistent with the SSR and is an entangled state for modes $\widehat{a}, \widehat{b}$. The Bloch vector for spin operators $\widehat{S}_x, \widehat{S}_y, \widehat{S}_z$ is given by (see [7])

$$\langle \widehat{S}_x \rangle = N\frac{\pi}{8}\cos\theta_p, \qquad \langle \widehat{S}_y \rangle = -N\frac{\pi}{8}\sin\theta_p,$$
$$\langle \widehat{S}_z \rangle = 0 \tag{66}$$

but the covariance matrix (see equation (178) in [7]) is non-diagonal. Spin operator properties of the relative phase state are set out in appendix O.

3.7.1. New spin operators. It is more instructive to consider spin squeezing in terms of new spin operators $\widehat{J}_x, \widehat{J}_y, \widehat{J}_z$ for which the covariance matrix is diagonal. The new spin operators are related to the original spin operators via

$$\widehat{J}_x = \widehat{S}_z,$$
$$\widehat{J}_y = \sin\theta_p\, \widehat{S}_x + \cos\theta_p\, \widehat{S}_y,$$
$$\widehat{J}_z = -\cos\theta_p\, \widehat{S}_x + \sin\theta_p\, \widehat{S}_y \tag{67}$$

corresponding to the transformation in equation (3) with Euler angles $\alpha = -\pi + \theta_p$, $\beta = -\pi/2$ and $\gamma = -\pi$.

3.7.2. Bloch vector and covariance matrix. The Bloch vector and covariance matrix for spin operators $\widehat{J}_x, \widehat{J}_y, \widehat{J}_z$ are given by (see equations (180) and (181) in [7]—note that the $C(\widehat{J}_y, \widehat{J}_y)$ element is incorrect in equation (181), see second of [7])

$$\langle \widehat{J}_x \rangle = 0, \qquad \langle \widehat{J}_y \rangle = 0, \qquad \langle \widehat{J}_z \rangle = -N\frac{\pi}{8} \tag{68}$$

and

$$\begin{bmatrix} C(\widehat{J}_x,\widehat{J}_x) & C(\widehat{J}_x,\widehat{J}_y) & C(\widehat{J}_x,\widehat{J}_z) \\ C(\widehat{J}_y,\widehat{J}_x) & C(\widehat{J}_y,\widehat{J}_y) & C(\widehat{J}_y,\widehat{J}_z) \\ C(\widehat{J}_z,\widehat{J}_y) & C(\widehat{J}_z,\widehat{J}_y) & C(\widehat{J}_z,\widehat{J}_z) \end{bmatrix} \doteqdot \begin{bmatrix} \frac{1}{12}N^2 & 0 & 0 \\ 0 & \frac{1}{4}+\frac{1}{8}\ln N & 0 \\ 0 & 0 & \left(\frac{1}{6}-\frac{\pi^2}{64}\right)N^2 \end{bmatrix} \quad N \gg 1. \quad (69)$$

With $\langle \Delta\widehat{J}_x^2\rangle = \frac{1}{12}N^2$, $\langle \Delta\widehat{J}_y^2\rangle = \frac{1}{4}+\frac{1}{8}\ln N$ and $\langle \Delta\widehat{J}_z^2\rangle = \left(\frac{1}{6}-\frac{\pi^2}{64}\right)N^2$ and the only non-zero Bloch vector component being $\langle \widehat{J}_z\rangle = -N\frac{\pi}{8}$ it is easy to see that $\langle \Delta\widehat{J}_x^2\rangle\langle \Delta\widehat{J}_y^2\rangle \geqslant \frac{1}{4}|\langle \widehat{J}_z\rangle|^2$ as required by the Heisenberg Uncertainty Principle. The principal spin fluctuations in both $\widehat{J}_x$ and $\widehat{J}_z$ are comparable to the length of the Bloch vector and no spin squeezing occurs in either of these components. However, spin squeezing occurs in that $\widehat{J}_y$ is squeezed with respect to $\widehat{J}_x$, since $\langle \Delta\widehat{J}_y^2\rangle$ only increases as $\frac{1}{8}\ln N$ while $\frac{1}{2}|\langle \widehat{J}_z\rangle|$ increases as $\frac{\pi}{16}N$ for large N. Hence the relative phase state satisfies the test in equation (37) to demonstrate entanglement for modes $\widehat{c}$, $\widehat{d}$. Here $\sqrt{\langle \Delta\widehat{J}_y^2\rangle}/|\langle \widehat{J}_z\rangle| \sim \sqrt{\ln N}/N$ which indicates that the Heisenberg limit is being reached.

Note that *none* of the old spin components are spin squeezed. As shown in [7] $\langle \Delta\widehat{S}_x^2\rangle = \left(\frac{1}{6}-\frac{\pi^2}{64}\right)\cos^2\theta_p\, N^2$, $\langle \Delta\widehat{S}_y^2\rangle = \left(\frac{1}{6}-\frac{\pi^2}{64}\right)\sin^2\theta_p\, N^2$ and $\langle \Delta\widehat{S}_z^2\rangle = \frac{1}{12}N^2$, along with $\langle \widehat{S}_x\rangle = N\frac{\pi}{8}\cos\theta_p$ $\langle \widehat{S}_y\rangle = -N\frac{\pi}{8}\sin\theta_p$ $\langle \widehat{S}_z\rangle = 0$. All variances are of order N^2 while the non-zero means are only of order N. Hence spin squeezing in one of the principal spin operators does *not* imply spin squeezing in any of the original spin operators. This is relevant to spin squeezing tests for entanglement of the *original* modes.

3.7.3. New modes operators. To confirm that the relative phase state is in fact an entangled state for modes $\widehat{c}$, $\widehat{d}$ as well as for the original modes $\widehat{a}$, $\widehat{b}$, we note that the new mode operators $\widehat{c}$, $\widehat{d}$ are given in in equation (12) with Euler angles $\alpha = -\pi + \theta_p$, $\beta = -\pi/2$ and $\gamma = -\pi$. The old mode operators are given in equation (14) and with these Euler angles we have

$$\widehat{a} = -\exp\left(\frac{1}{2}\mathrm{i}\theta_p\right)\frac{1}{\sqrt{2}}(\widehat{c}-\widehat{d}),$$
$$\widehat{b} = -\exp\left(-\frac{1}{2}\mathrm{i}\theta_p\right)\frac{1}{\sqrt{2}}(\widehat{c}+\widehat{d}). \quad (70)$$

This enables us to write the phase state in terms of new mode operators $\widehat{c}$, $\widehat{d}$ as

$$\left|\frac{N}{2},\theta_p\right\rangle = \frac{1}{\sqrt{N+1}}\left(\frac{-1}{\sqrt{2}}\right)^N \sum_{k=-N/2}^{N/2}\sum_{r=-N/4+k/2}^{N/4-k/2}\sum_{s=-N/4-k/2}^{N/4+k/2} \times \frac{1}{\sqrt{(N/2-k)!}}\frac{1}{\sqrt{(N/2+k)!}}(-1)^{N/4-k/2+r} \times \frac{(N/2-k)!}{(N/4-k/2-r)!(N/4-k/2+r)!} \times \frac{(N/2+k)!}{(N/4+k/2-s)!(N/4+k/2+s)!} \times (\widehat{c}^\dagger)^{N/2-(r+s)}(\widehat{d}^\dagger)^{N/2+(r+s)}|0\rangle. \quad (71)$$

We see that the expression does not depend explicitly on the relative phase θ_p when written in terms of the new mode unnormalized Fock states $(\widehat{c}^\dagger)^{N/2-(r+s)}(\widehat{d}^\dagger)^{N/2+(r+s)}|0\rangle$. This pure state is a linear combination of product states of the form $|N/2-m\rangle_c \otimes |N/2+m\rangle_d$ for various m—each of which is an N boson state and an eigenstate for $\widehat{J}_z$ with eigenvalue m, and therefore is an entangled state for modes $\widehat{c}$, $\widehat{d}$ which is compatible with the global SSR. Note that there cannot just be a single term m involved, otherwise the variance for $\widehat{J}_z$ would be zero instead of $\left(\frac{1}{6}-\frac{\pi^2}{64}\right)N^2$. We will return to the relative phase state again in section 4.1.

4. Other spin operator tests for entanglement

In this section we examine a number of previously stated entanglement tests involving spin operators. It turns out that *many* of the tests do confirm entanglement for massive bosons according to the SSR and symmetrization principle compliant definition as it is defined here, though not always for the reasons given in their original proofs. Importantly, in some cases for massive bosons the tests can be made more general.

There are a number of inequalities involving the *spin operators* that have previously been derived for testing whether a state for a system of identical bosons is entangled. These are *not* always associated with criteria for spin squeezing—which involve the variances and mean values of the spin operators. Also, some of these inequalities are *not* based on the requirement that the density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ in the expression for a non-entangled state conform to the SSR that prohibits quantum superpositions of single mode states with differing numbers of bosons (which was invoked because they represent possible quantum states for the separate modes—*local particle number SSR compliance*). Only generic quantum properties of the sub-system density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ were used in the derivations. In contrast, our results are based in effect on a *stricter criterion* as to what constitutes a *separable state*, so of course we obtain *new* entanglement tests. However, entanglement tests which were based on *not requiring* SSR compliance for $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ will *also* confirm entanglement when SSR compliance is *required*. This outcome occurs in the section 4.1 in the case of the Hillery spin variance entanglement test. It also occurs in section 4.2 for the entanglement test in (101) involving spin operators for two mode sub-systems, in section 4.3 for the entanglement test in (104) involving mean values of powers of local spin operators, and in two entanglement tests (110) and (111) in section 4.5 that involve variances of two mode spin operators.

Other entanglement tests have been proposed whose proofs were based on forms of the density operator for non-entangled states that are *not* consistent with the *symmetrization principle*. The sub-systems were regarded as labeled individual particles, and strictly speaking, this should only apply to systems of *distinguishable* particles. These include the spin squeezing in the total spin operator $\widehat{S}_z$ test (108) in section 4.4. In appendix H we show that the original proof in [14] can be modified to treat *identical* particles but now with distinguishable pairs of modes as the sub-systems, but the proof requires that the separable states are *restricted* to one boson per mode pair. However, in section 3.1 we have already shown that for two mode systems in which SSR compliance applies spin squeezing in $\widehat{S}_z$ demonstrates two mode entanglement. Also, in appendix D we show that in multi-mode cases modes associated with two different internal (hyperfine) components, that spin squeezing in $\widehat{S}_z$ also shows entanglement occurs in two situations—one where there are *two* sub-systems each just consisting of modes associated with the same internal component (Case 1), the second where *each* mode counts as a separate sub-system (Case 2). Thus the spin squeezing in $\widehat{S}_z$ test does demonstrate entanglement for identical massive bosons, though not for the reasons given in the original proof. These new SSR compliant proofs now confirm the spin squeezing in $\widehat{S}_z$ as a valid test for entanglement in two component or two mode BECs.

4.1. Hillery et al 2006

4.1.1. Hillery spin variance entanglement test. An entanglement test in which local particle number SSR compliance is ignored is presented in the paper by Hillery and Zubairy [22] entitled 'Entanglement conditions for two-mode states'. The paper actually dealt with EM field modes, and the density operators $\hat{\rho}_R^A, \hat{\rho}_R^B$ for photon modes allowed for coherences between states with differing photon numbers. A discussion of SSR for the case of photons is presented in paper I, in appendix L. Hence the conditions in equation (16) were not applied.

We will now derive the Hillery spin variance inequalities involving $\langle\Delta\widehat{S}_x^2\rangle$, $\langle\Delta\widehat{S}_y^2\rangle$ by applying a similar treatment to that in section 3.1, but now ignoring local particle number SSR compliance. It is found that for the original spin operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ and modes $\widehat{a}$ and $\widehat{b}$

$$\begin{aligned}\langle\widehat{S}_x^2\rangle_R &= \frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R)\\ &\quad + \frac{1}{4}(\langle(\widehat{b}^\dagger)^2\rangle_R\langle(\widehat{a})^2\rangle_R + \langle(\widehat{b})^2\rangle_R\langle(\widehat{a}^\dagger)^2\rangle),\\ \langle\widehat{S}_y^2\rangle_R &= \frac{1}{4}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R)\\ &\quad - \frac{1}{4}(\langle(\widehat{b}^\dagger)^2\rangle_R\langle(\widehat{a})^2\rangle_R + \langle(\widehat{b})^2\rangle_R\langle(\widehat{a}^\dagger)^2\rangle)\end{aligned} \tag{72}$$

where terms such as $\langle(\widehat{b}^\dagger)^2\rangle_R$ and $\langle(\widehat{a})^2\rangle_R$ previously shown to be zero have been retained. Note that in [22] the operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ constructed from the EM field mode operators as in equation (1)) would be related to Stokes parameters Hence

$$\begin{aligned}&\langle\widehat{S}_x^2\rangle_R + \langle\widehat{S}_y^2\rangle_R\\ &= \frac{1}{2}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + (\langle\widehat{a}^\dagger\widehat{a}\rangle_R\langle\widehat{b}^\dagger\widehat{b}\rangle_R)\\ &= \frac{1}{2}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R(\langle\widehat{a}^\dagger\widehat{a}\rangle_R + 1) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + 1)),\end{aligned} \tag{73}$$

where the terms $\langle(\widehat{b}^\dagger)^2\rangle_R$, ..., $\langle(\widehat{a}^\dagger)^2\rangle$ cancel out. This is the same as before.

However,

$$\begin{aligned}\langle\widehat{S}_x\rangle_R &= \frac{1}{2}(\langle\widehat{b}^\dagger\rangle_R\langle\widehat{a}\rangle_R + \langle\widehat{a}^\dagger\rangle_R\langle\widehat{b}\rangle_R),\\ \langle\widehat{S}_y\rangle_R &= \frac{1}{2i}(\langle\widehat{b}^\dagger\rangle_R\langle\widehat{a}\rangle_R - \langle\widehat{a}^\dagger\rangle_R\langle\widehat{b}\rangle_R)\end{aligned} \tag{74}$$

is now non-zero, since the previously zero terms have again been retained. Hence

$$\langle\widehat{S}_x\rangle_R^2 + \langle\widehat{S}_y\rangle_R^2 = \langle\widehat{b}^\dagger\rangle_R\langle\widehat{b}\rangle_R\langle\widehat{a}^\dagger\rangle_R\langle\widehat{a}\rangle_R \tag{75}$$

so that we now have

$$\begin{aligned}&\langle\Delta\widehat{S}_x^2\rangle_R + \langle\Delta\widehat{S}_y^2\rangle_R\\ &= \frac{1}{2}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R(\langle\widehat{a}^\dagger\widehat{a}\rangle_R + 1) + \frac{1}{2}(\langle\widehat{a}^\dagger\widehat{a}\rangle_R(\langle\widehat{b}^\dagger\widehat{b}\rangle_R) + 1)\\ &\quad - \langle\widehat{b}^\dagger\rangle_R\langle\widehat{b}\rangle_R\langle\widehat{a}\rangle_R\langle\widehat{a}^\dagger\rangle_R\langle\widehat{a}\rangle_R\\ &= \frac{1}{2}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R)\\ &\quad + (\langle\widehat{b}^\dagger\widehat{b}\rangle_R(\langle\widehat{a}^\dagger\widehat{a}\rangle_R - |\langle\widehat{a}\rangle_R|^2|\langle\widehat{b}^\dagger\rangle_R|^2).\end{aligned} \tag{76}$$

But from the Schwarz inequality—which is based on $\langle(\widehat{a}^\dagger - \langle\widehat{a}^\dagger\rangle)(\widehat{a} - \langle\widehat{a}\rangle)\rangle \geqslant 0$ for any state

$$|\langle\widehat{a}\rangle_R|^2 \leqslant \langle\widehat{a}^\dagger\widehat{a}\rangle_R \qquad |\langle\widehat{b}\rangle_R|^2 \leqslant \langle\widehat{b}^\dagger\widehat{b}\rangle_R \tag{77}$$

so that

$$\langle\Delta\widehat{S}_x^2\rangle_R + \langle\Delta\widehat{S}_y^2\rangle_R \geqslant \frac{1}{2}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) \tag{78}$$

and thus from equation (20) it follows that for a general non entangled state

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle \geqslant \sum_R P_R\frac{1}{2}(\langle\widehat{n}_b\rangle_R + \langle\widehat{n}_a\rangle_R). \tag{79}$$

However, half the expectation value of the number operator is

$$\frac{1}{2}\langle\widehat{N}\rangle = \frac{1}{2}\langle(\widehat{n}_a + \widehat{n}_b)\rangle = \sum_R P_R\frac{1}{2}(\langle\widehat{n}_b\rangle_R + \langle\widehat{n}_a\rangle_R) \tag{80}$$

so for a non-entangled state

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle \geqslant \frac{1}{2}\langle\widehat{N}\rangle. \tag{81}$$

This inequality for non-entangled states is given in [22] (see their equation (3)). The above proof was based on *not* invoking the SSR requirements for separable states that we apply in this paper.

The Hillery spin variance test can also be applied in multi-mode situations, where the spin operators are defined as in appendix A and there are three cases involving different choices of sub-systems (see appendix D). The Hillery spin variance test applies for Cases 1 and 2, but not for Case 3. The derivation of the Hillary spin variance test for the multi-mode situation is presented in appendix E.

4.1.2. Validity of Hillery test for local SSR compliant non-entangled states. However, it is interesting that the inequality (81) can be more readily derived from the definition of entangled states used in the present paper—which is based on local particle number SSR compliance for separable states. We would then find that $\langle\widehat{S}_x\rangle_R = \langle\widehat{S}_y\rangle_R = 0$ and hence

$$\langle\Delta\widehat{S}_x^2\rangle_R + \langle\Delta\widehat{S}_y^2\rangle_R = \frac{1}{2}(\langle\widehat{b}^\dagger\widehat{b}\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R) + (\langle\widehat{b}^\dagger\widehat{b}\rangle_R(\langle\widehat{a}^\dagger\widehat{a}\rangle_R) \tag{82}$$

instead of equation (76). Since the term $\langle\widehat{b}^\dagger\widehat{b}\rangle_R(\langle\widehat{a}^\dagger\widehat{a}\rangle_R$ is always positive we find after applying equation (20) that

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle \geqslant \frac{1}{2}\langle\widehat{N}\rangle \tag{83}$$

which is the same as in equation (81). Hence, finding that $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle < \frac{1}{2}\langle\widehat{N}\rangle$ would show that the state was entangled, irrespective of whether or not entanglement is defined in terms of non-physical nonentangled states.

Thus, the *Hillery spin variance* entanglement test [22] is that *if*

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle < \frac{1}{2}\langle\widehat{N}\rangle \tag{84}$$

then the state is an entangled state of modes $\widehat{a}$ and $\widehat{b}$. This test is still used in recent papers, for example [23, 24] which deal with the entanglement of sub-systems each consisting of single modes $\widehat{a}, \widehat{b}$ for a double well situation (in these papers $\widehat{S}_x \to \widehat{J}_{AB}^X$, $\widehat{S}_y \to -\widehat{J}_{AB}^Y$, $\widehat{S}_z \to -\widehat{J}_{AB}^Z$).

4.1.3. Non-applicable entanglement test involving $|\langle\widehat{S}_z\rangle|$. Previously we had found for a general non-entangled state that is based on physically valid density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$

$$\langle\Delta\widehat{S}_x^2\rangle - \frac{1}{2}|\langle\widehat{S}_z\rangle| \geqslant 0,$$
$$\langle\Delta\widehat{S}_y^2\rangle - \frac{1}{2}|\langle\widehat{S}_z\rangle| \geqslant 0 \tag{85}$$

so that the sum of the variances satisfies the inequality

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle \geqslant |\langle\widehat{S}_z\rangle|. \tag{86}$$

This is another valid inequality required for a non-entangled state as defined in the present paper. It follows that if only physical states $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ are allowed, the related *entanglement test* involving $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle$ would be

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle < |\langle\widehat{S}_z\rangle|. \tag{87}$$

For *any* quantum state we have

$$|\langle\widehat{S}_z\rangle| = \frac{1}{2}|(\langle\widehat{n}_b\rangle - \langle\widehat{n}_a\rangle)| \leqslant \frac{1}{2}(\langle\widehat{n}_b\rangle + \langle\widehat{n}_a\rangle) = \frac{1}{2}\langle\widehat{N}\rangle \tag{88}$$

which means that it is now required that $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle$ be less than a quantity that is *smaller* than in the criterion in (81).

However, it should be noted (see (7)) that *all* states, entangled or otherwise, satisfy the inequality

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle \geqslant |\langle\widehat{S}_z\rangle| \tag{89}$$

so the inequality in (86)—though true, is of no use in establishing whether a state is entangled in the terms of the meaning of entanglement in the present paper. There are *no* quantum states, entangled or otherwise that satisfy the proposed entanglement test given in equation (87). This general result was stated by Hillery *et al* [22]. To show this we have

$$\langle(\Delta\widehat{S}_x - \mathrm{i}\lambda\Delta\widehat{S}_y)^\dagger(\Delta\widehat{S}_x + \mathrm{i}\lambda\Delta\widehat{S}_y)\rangle \geqslant 0, \tag{90}$$

$$\langle\Delta\widehat{S}_x^2\rangle + \lambda\langle\widehat{S}_z\rangle + \lambda^2\langle\Delta\widehat{S}_y^2\rangle \geqslant 0 \tag{91}$$

for all real λ. The condition that this function of λ is never negative gives the Heisenberg uncertainty principle $\langle\Delta\widehat{S}_x^2\rangle\langle\Delta\widehat{S}_y^2\rangle \geqslant \frac{1}{4}|\langle\widehat{S}_z\rangle|^2$ and (89) follows from taking $\lambda = 1$ and $\lambda = -1$. Even spin squeezed states with $\langle\Delta\widehat{S}_x^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle|$ still have $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle \geqslant |\langle\widehat{S}_z\rangle|$, so it is *never* found that $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle < |\langle\widehat{S}_z\rangle|$ and hence this latter inequality *cannot* used as a test for entanglement.

Fortunately—as we have seen, showing that spin squeezing occurs via *either* $\langle\Delta\widehat{S}_x^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle|$ *or* $\langle\Delta\widehat{S}_y^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle|$ is sufficient to establish that the state is an entangled state for modes $\widehat{a}, \widehat{b}$, with analogous results if principle spin operators are considered. Applying the Hillery *et al* entanglement test in equation (84) involving $\frac{1}{2}\langle\widehat{N}\rangle$ is also a valid entanglement test, but is usually *less stringent* than the spin squeezing test involving either $\langle\Delta\widehat{S}_x^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle|$ *or* $\langle\Delta\widehat{S}_y^2\rangle < \frac{1}{2}|\langle\widehat{S}_z\rangle|$. For the Hillery *et al* entanglement test to be satisfied at least one of $\langle\Delta\widehat{S}_x^2\rangle$ or $\langle\Delta\widehat{S}_y^2\rangle$ is required to be less than $\frac{1}{2}\langle\widehat{N}\rangle$, whereas for the spin squeezing test to apply at least one of $\langle\Delta\widehat{S}_x^2\rangle$ or $\langle\Delta\widehat{S}_y^2\rangle$ must be less than $\frac{1}{2}|\langle\widehat{S}_z\rangle|$. The quantity $\frac{1}{2}|\langle\widehat{S}_z\rangle|$ is likely to be smaller than $\frac{1}{2}\langle\widehat{N}\rangle$—for example the Bloch vector may lie close to the *xy* plane, so a greater degree of reduction in spin fluctuation is needed to satisfy the spin squeezing test for entanglement.

However, this is not always the case as the example of the *relative phase state* discussed in section 3.7 shows. The results in the current section can easily be modified to apply to new spin operators $\widehat{J}_x$, $\widehat{J}_y$, $\widehat{J}_z$, with entanglement being considered for new modes $\widehat{c}$ and $\widehat{d}$. The Hillery *et al* [22] entanglement test then becomes

$$\langle \Delta \hat{J}_x^2 \rangle + \langle \Delta \hat{J}_y^2 \rangle < \frac{1}{2} \langle \hat{N} \rangle. \quad (92)$$

In the case of the relative phase eigenstate we have from equation (69) that $\langle \Delta \hat{J}_x^2 \rangle + \langle \Delta \hat{J}_y^2 \rangle = \frac{1}{12}N^2 + \frac{1}{4} + \frac{1}{8}\ln N \approx \frac{1}{12}N^2$ for large N. This clearly exceeds $\frac{1}{2}\langle \hat{N} \rangle = \frac{1}{2}N$, so the Hillery *et al* [22] test for entanglement fails. On the other hand, as we have seen in section 3.7 $\langle \Delta \hat{J}_y^2 \rangle < \frac{1}{2}|\langle \hat{J}_z \rangle| \approx \frac{\pi}{16}N$, so the spin squeezing test is satisfied for this entangled state of modes $\hat{c}$ and $\hat{d}$.

4.2. He et al 2012

In two papers dealing with EPR entanglement He *et al* [23, 25] a *four mode* system associated with a double well potential is considered. In the left well 1 there are two localized modes with annihilation operators $\hat{a}_1$, $\hat{b}_1$ and in the right well 2 there are two localized modes with annihilation operators $\hat{a}_2$, $\hat{b}_2$. The modes in each well are associated with two different internal states A and B. Note that we use a different notation to [23, 25]. This four mode system provides for the possibility of entanglement of *two sub-systems* each consisting of *pairs of modes*. We can therefore still consider *bipartite* entanglement however. With four modes there are three different choices of such sub-systems but perhaps the most interesting from the point of view of entanglement of spatially separated modes—and hence implications for EPR entanglement—would be to have the two *left well* modes $\hat{a}_1$, $\hat{b}_1$ as sub-system 1 and the two *right well* modes $\hat{a}_2$, $\hat{b}_2$ as sub-system 2. This is an example of the general Case 3 considered for multi-modes in section D.1. Consistent with the requirement that the sub-system density operators $\hat{\rho}_R^{ab(1)}$, $\hat{\rho}_R^{ab(2)}$ conform to the symmetrization principle and the SSR, these density operators will not in general represent separable states for their single mode sub-systems $\hat{a}_1$, $\hat{b}_1$ or $\hat{a}_2$, $\hat{b}_2$—and may even be entangled states. As a result when considering *non-entangled* states for the pair of sub-systems 1 and 2 we now have

$$\langle (\hat{a}_i^\dagger \hat{b}_i)^n \rangle_{ab(i)} = \mathrm{Tr}(\hat{\rho}_R^{ab(i)} (\hat{a}_i^\dagger \hat{b}_i)^n) \neq 0, \qquad i = 1, 2 \quad (93)$$

in general. In this case where the sub-systems are *pairs* of modes the spin squeezing entanglement tests as in equations (47)–(49) for sub-systems consisting of *single* modes cannot be applied, as explained for Case 3 in section D.4 unless there is only one boson in each sub-system. Nevertheless, there are still tests of bipartite entanglement involving spin operators. We next examine entanglement tests in [23, 25] to see if any changes occur when we invoke the definition of entanglement based on SSR compliance.

4.2.1. Spin operator tests for entanglement. There are numerous choices for defining spin operators, but the most useful would be the *local spin operators* for each well [23] defined by

$$\begin{aligned}
&\hat{S}_x^1 = (\hat{b}_1^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{b}_1)/2, \qquad \hat{S}_y^1 = (\hat{b}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{b}_1)/2i,\\
&\hat{S}_z^1 = (\hat{b}_1^\dagger \hat{b}_1 - \hat{a}_1^\dagger \hat{a}_1)/2,\\
&\hat{S}_x^2 = (\hat{b}_2^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{b}_2)/2, \qquad \hat{S}_y^2 = (\hat{b}_2^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{b}_2)/2i,\\
&\hat{S}_z^2 = (\hat{b}_2^\dagger \hat{b}_2 - \hat{a}_2^\dagger \hat{a}_2)/2.
\end{aligned} \quad (94)$$

These satisfy the usual angular momentum commutation rules and those or the different wells commute. The squares of the local vector spin operators are related to the total number operators $\hat{N}_1 = \hat{b}_1^\dagger \hat{b}_1 + \hat{a}_1^\dagger \hat{a}_1$ and $\hat{N}_2 = \hat{b}_2^\dagger \hat{b}_2 + \hat{a}_2^\dagger \hat{a}_2$ as $\sum_\alpha (\hat{S}_\alpha^1)^2 = (\hat{N}_1/2)(\hat{N}_1/2 + 1)$ and $\sum_\alpha (\hat{S}_\alpha^2)^2 = (\hat{N}_2/2)(\hat{N}_2/2 + 1)$. The *total spin operators* are

$$\hat{S}_\alpha = \hat{S}_\alpha^1 + \hat{S}_\alpha^2, \qquad \alpha = x, y, z \quad (95)$$

and these satisfy the usual angular momentum commutation rules. Hence there may be cases of spin squeezing, but these do not in general provide entanglement tests.

For the local spin operators we have in general

$$\begin{aligned}
&\langle \hat{S}_\alpha^1 \rangle_{ab(1)} = \mathrm{Tr}(\hat{\rho}_R^{ab(1)}\, \hat{S}_\alpha^1) \neq 0,\\
&\langle \hat{S}_\alpha^2 \rangle_{ab(2)} = \mathrm{Tr}(\hat{\rho}_R^{ab(2)}\, \hat{S}_\alpha^2) \neq 0, \qquad \alpha = x, y, z
\end{aligned} \quad (96)$$

based on (93), and applying (19) we see that *in general* $\langle \hat{S}_\alpha \rangle \neq 0$ for separable states. Thus the Bloch vector test for entanglement does not apply.

Furthermore, there is no spin squeezing test either. Following a similar approach as in section 3 we can obtain the following inequalities for separable states of the sub-systems 1 and 2

$$\begin{aligned}
&\langle \Delta \hat{S}_x^2 \rangle - \frac{1}{2}|\langle \hat{S}_z \rangle|\\
&\quad \geqslant \sum_R P_R(\langle (\Delta \hat{S}_x^1)^2 \rangle_R - \frac{1}{2}|\langle \hat{S}_z^1 \rangle_R|)\\
&\qquad + \sum_R P_R(\langle (\Delta \hat{S}\ {}_x^2)^2 \rangle_R - \frac{1}{2}|\langle \hat{S}_z^2 \rangle_R|),
\end{aligned} \quad (97)$$

$$\begin{aligned}
&\langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2}|\langle \hat{S}_z \rangle|\\
&\quad \geqslant \sum_R P_R(\langle (\Delta \hat{S}_y^1)^2 \rangle_R - \frac{1}{2}|\langle \hat{S}_z^1 \rangle_R|)\\
&\qquad + \sum_R P_R(\langle (\Delta \hat{S}_y^2)^2 \rangle_R - \frac{1}{2}|\langle \hat{S}_z^2 \rangle_R|).
\end{aligned} \quad (98)$$

Similar inequalities can be obtained for other pairs of spin operators. In neither case can we state that the right sides are always non-negative, For example, each $\hat{\rho}_R^{ab(1)}$ may be a spin squeezed state for $\hat{S}_x^1$ versus $\hat{S}_y^1$ and each $\hat{\rho}_R^{ab(2)}$ may be a spin squeezed state for $\hat{S}_x^2$ versus $\hat{S}_y^2$. In this case the right side of the first inequality is a negative quantity, so we cannot conclude that the total $\hat{S}_x$ is *not* squeezed versus $\hat{S}_y$ for *all* separable states. As the $\hat{\rho}_R^{ab(1)}$ and $\hat{\rho}_R^{ab(2)}$ can be chosen independently we see that separable states for the sub-systems 1 and 2 may *be* spin squeezed, so the presence of spin squeezing in a *total* spin operator is not a test for bipartite entanglement in this four mode system. This does not of course preclude tests for bipartite entanglement involving spin operators, as we will now see.

In section 2.8 of paper 1 it was shown that $|\langle\widehat{\Omega}_A^\dagger\widehat{\Omega}_B\rangle|^2 \leqslant \langle\widehat{\Omega}_A^\dagger\,\widehat{\Omega}_A\,\widehat{\Omega}_B^\dagger\,\widehat{\Omega}_B\rangle$ for a non-entangled state of general sub-systems A *and* B, so with $\widehat{\Omega}_A \to \widehat{S}_-^1 = \widehat{S}_x^1 - \mathrm{i}\widehat{S}_y^1$ and $\widehat{\Omega}_B \to \widehat{S}_-^2 = \widehat{S}_x^2 - \mathrm{i}\widehat{S}_y^2 = (\widehat{S}_+^2)^\dagger$ to give

$$|\langle\widehat{S}_+^1\,\widehat{S}_-^2\rangle|^2 \leqslant \langle\widehat{S}_+^1\,\widehat{S}_-^1\,\widehat{S}_+^2\,\widehat{S}_-^2\,\rangle \tag{99}$$

for a non-entangled state of sub-systems 1 and 2. For the non-entangled state of these two sub-systems we have

$$\langle\widehat{S}_+^1\,\widehat{S}_-^2\rangle = \sum_R P_R\,\langle\widehat{S}_+^1\rangle_{ab(1)}^R\,\langle\widehat{S}_-^2\rangle_{ab(2)}^R \tag{100}$$

which in general is non-zero from equation (96).

Hence a valid *entanglement test* involving *spin operators* for sub-systems 1 and 2—*each* consisting of *two modes* localized in each well exists, so if

$$|\langle\widehat{S}_+^1\,\widehat{S}_-^2\rangle|^2 > \langle\widehat{S}_+^1\,\widehat{S}_-^1\,\widehat{S}_+^2\,\widehat{S}_-^2\,\rangle \tag{101}$$

then the two sub-systems are entangled. A similar conclusion is stated in [23], where the criterion was predicted to be satisfied for four mode two well BEC systems. This test for entanglement involves the local spin operators, though it is not then the same as spin squeezing criteria. It is referred to there as *spin entanglement*. Other similar tests may be obtained via different choices of $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$.

4.3. Raymer et al 2003

In a paper also dealing with bipartite entanglement where the sub-systems each consist of two modes, Raymer *et al* [26] derive entanglement tests involving spin operators for the sub-systems defined in (94).

The *Raymer* entanglement test is that *if*

$$\begin{aligned}&\langle\Delta(\alpha\widehat{\Omega}_A + \beta\widehat{\Omega}_B)^2\rangle + \langle\Delta(\alpha\widehat{\Lambda}_A - \beta\widehat{\Lambda}_B)^2\rangle\\ &\quad < \alpha^2|\langle\widehat{\Theta}_A\rangle| \; + \beta^2|\langle\widehat{\Theta}_B\rangle|,\end{aligned} \tag{102}$$

where α, β are real and $\widehat{\Theta}_A$, $\widehat{\Theta}_B$ are given by the commutators

$$[\widehat{\Omega}_A, \widehat{\Lambda}_A] = \mathrm{i}\widehat{\Theta}_A \qquad [\widehat{\Omega}_B, \widehat{\Lambda}_B] = \mathrm{i}\widehat{\Theta}_B \tag{103}$$

then the state is entangled. The proof is given in appendix F.

We now choose $\widehat{\Omega}_A = \widehat{S}_x^1$, $\widehat{\Omega}_B = \widehat{S}_x^2$, $\widehat{\Lambda}_A = \widehat{S}_y^1$ and $\widehat{\Lambda}_B = \widehat{S}_y^2$ as in equation (94) along with $\alpha = \beta = 1$ Here $\widehat{\Theta}_A = \widehat{S}_z^1$ and $\widehat{\Theta}_B = \widehat{S}_z^2$. Here sub-systems $A = 1$, $B = 2$ consist of modes $\widehat{a}_1$, $\widehat{b}_1$ and $\widehat{a}_2$, $\widehat{b}_2$ respectively. Hence if we have

$$\langle\Delta(\widehat{S}_x^1 + \widehat{S}_x^2)^2\rangle + \langle\Delta(\widehat{S}_y^1 - \widehat{S}_y^2)^2\rangle < |\langle\widehat{S}_z^1\rangle| \; + |\langle\widehat{S}_z^2\rangle| \tag{104}$$

then *bipartite entanglement* is established. Note that this test did not require local particle number SSR compliance, but still will apply if this is invoked. Other tests involving a cyclic interchange of x, y, z can also be established, as can other tests where the signs within the left terms are replaced by $(-, +)$, $(+, +)$, $(-, -)$ via appropriate choices of α, β. These tests involve mean values of powers of *local* spin operators. Similar to tests in sections 4.1 and 4.2, this test also does not require SSR compliance.

4.4. Sørensen et al 2001

4.4.1. Sørensen spin squeezing entanglement test. In a paper entitled 'many-particle entanglement with Bose–Einstein condensates' Sørensen *et al* [14] consider the implications for spin squeezing for non-entangled states of the form

$$\widehat{\rho} = \sum_R P_R\;\widehat{\rho}_R^1 \otimes \widehat{\rho}_R^2 \otimes \widehat{\rho}_R^3 \otimes ..., \tag{105}$$

where $\widehat{\rho}_R^i$ is a density operator for particle i. As discussed previously, a density operator of this general form is not consistent with the symmetrization principle—having separate density operators $\widehat{\rho}_R^i$ for specific particles i in an identical particle system (such as for a BEC) is not compatible with the indistinguishability of such particles. It is modes that are distinguishable, not identical particles, so the basis for applying their results to systems of identical bosons is flawed. However, they derive an inequality for the spin variance $\langle\Delta\widehat{S}_z^2\rangle$

$$\langle\Delta\widehat{S}_z^2\rangle \geqslant \frac{1}{N}(\langle\widehat{S}_x\rangle^2 + \langle\widehat{S}_y\rangle^2) \tag{106}$$

that applies in the case of non-entangled states. Key steps in their derivation are stated in the appendix to [14], but as the justification of these steps is not obvious for completeness the full derivation is given in appendix G of the present paper. This inequality (106) establishes that if

$$\xi^2 = \frac{\langle\Delta\widehat{S}_z^2\rangle}{(\langle\widehat{S}_x\rangle^2 + \langle\widehat{S}_y\rangle^2)} < \frac{1}{N} \tag{107}$$

then the state is entangled, so that if the condition for spin squeezing analogous to that in appendix B equation (201) is satisfied, then entanglement is required if spin squeezing for $\widehat{S}_z$ to occur. Spin squeezing is then a test for entanglement in terms of their definition of an entangled state.

If the Bloch vector is close to the Bloch sphere, for example with $\langle\widehat{S}_x\rangle = 0$ and $\langle\widehat{S}_y\rangle = N/2$ then the condition (107) is equivalent to

$$\langle\Delta\widehat{S}_z^2\rangle < \frac{1}{2}|\langle\widehat{S}_y\rangle| \tag{108}$$

which is the condition for squeezing in $\widehat{S}_z$ compared to $\widehat{S}_x$. Spin squeezing is then a test for entanglement in terms of their definition of an entangled state. Note that the condition (108) requires the Bloch vector to be in the xy plane and close to the Bloch sphere of radius $N/2$. By comparison with appendix B (201) we see that the Sørensen *spin squeezing* test is that if there is squeezing in $\widehat{S}_z$ with respect to any spin component in the xy plane *and* the Bloch vector is close to the Bloch sphere, then the state is entangled.

As explained above, the proof of Sørensen *et al* really applies only when the individual spins are *distinguishable*. It is possible however to modify the work of Sørensen *et al* [14] to apply to a system of identical bosons in accordance with the symmetrization and SSRs if the index i is *re-interpreted* as specifying different modes, for example modes localized on *optical lattice* sites $i = 1, 2,.., n$ or distinct free space *momentum states* listed $i = 1, 2,.., n$. On each lattice site or for each momentum state there would be two modes a, b—for example associated with two different *internal states*—so the sub-system density operator $\hat{\rho}_R^i$ then applies to the two modes on site i. However the proof of (106) requires the $\hat{\rho}_R^i$ to be restricted to the case where there is exactly *one* identical boson on each site or in a momentum state. Such a localization process in position or momentum space has the effect of enabling the identical bosons to be treated *as if* they are distinguishable. Details are given in appendix H. A similar modification has been carried out by Hyllus *et al* [27].

However, as we have seen in section 3.2 it does in fact turn out for two mode systems of identical bosons that showing that $\hat{S}_z$ is spin squeezed compared to $\hat{S}_x$ or $\hat{S}_y$ *is* sufficient to prove that the quantum state is entangled. There are *no* restrictions either on the mean number of bosons occupying each mode. The proof is based on applying the requirement of local particle number SSR compliance to the separable states in the present case of massive bosons and treating modes (not particles) as sub-systems. In appendix D we have also shown that the same result applies to multi-mode situations in cases where the sub-systems are all single modes (Case 2) or where there are two sub-systems each containing all modes for a single component (Case 1). So the spin squeezing test is still *valid* for many particle BEC, though the justification is not as in the proof of Sørensen *et al* [14] (which was derived for systems of distinguishable particles), with each sub-system being a single two state particle.

4.5. Benatti et al 2011

In earlier work Toth and Gunhe [28] derived several spin operator based inequalities for separable states for two mode particle systems based on the assumption that the particles were *distinguishable*. As in equation (105), the density operator was not required to satisfy the symmetrization principle. Tests for entanglement involving the mean values and variances for two mode spin operators resulted. Subsequently, Benatti *et al* [29] considered whether these tests would still apply if the particles were *indistinguishable*. Their work involves considering states with N bosons.

For *separable* states they found (see equation (10)) that for three orthogonal spin operators $\hat{J}_{n1}$, $\hat{J}_{n2}$ and $\hat{J}_{n3}$

$$\langle \hat{J}_{n1}^2 \rangle + \langle \hat{J}_{n2}^2 \rangle + \langle \hat{J}_{n3}^2 \rangle \leqslant \frac{N(N+2)}{4} \tag{109}$$

from which it might be concluded that if the left side exceeded $N(N+2)/4$ then the state must be entangled. However, since $\hat{J}_{n1}^2 + \hat{J}_{n1}^2 + \hat{J}_{n1}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \hat{N}(\hat{N}+2)/4$ the left side is always equal to $N(N+2)/4$ for all states with N bosons, so no entanglement test results. This outcome is for similar reasons as for the failed entanglement test $\langle \Delta\hat{S}_x^2 \rangle + \langle \Delta\hat{S}_y^2 \rangle < |\langle \hat{S}_z \rangle|$ discussed in section 4.1.

Benatti *et al* [29] also showed that if

$$\langle \Delta\hat{J}_{n1}^2 \rangle + \langle \Delta\hat{J}_{n2}^2 \rangle + \langle \Delta\hat{J}_{n3}^2 \rangle < \frac{N}{2} \tag{110}$$

then the state must be entangled. This test is an extended form of the Hillery spin variance test (84). The test in (110) is quite useful in that it applies to any three orthogonal spin operators, though it would be harder to satisfy compared to the Hillery spin variance test because of the additional. $\langle \Delta\hat{S}_z^2 \rangle$ term. The proof is given in appendix I.

In addition, Benatti *et al* [29] also showed that if

$$(N-1)(\langle \Delta\hat{J}_{n1}^2 \rangle + \langle \Delta\hat{J}_{n2}^2 \rangle) - \langle \hat{J}_{n3}^2 \rangle < \frac{N(N-2)}{4} \tag{111}$$

then the state must be entangled. The proof is given in appendix I.

Finally, Benatti *et al* [29] considered a further inequality (see equation (12)) found to apply for *separable* states involving *distinguishable* particles in [28].

$$(\langle \hat{J}_{n1}^2 \rangle + \langle \hat{J}_{n2}^2 \rangle) - \frac{N}{2} - (N-1)\langle \Delta\hat{J}_{n3}^2 \rangle \leqslant 0. \tag{112}$$

However, as discussed in appendix I no entanglement test can be shown for identical particles.

Hence Benatti *et al* [29] have demonstrated two further entanglement tests (110) and (111) for two mode systems of identical particle that involve spin operators. Again, these tests do not involve invoking the local particle number SSR for separable states.

4.6. Sørensen and Mølmer 2001

In a paper entitled 'entanglement and extreme spin squeezing' Sørensen and Mølmer [30] first consider the limits imposed by the Heisenberg uncertainty principle on the variance $\langle \Delta\hat{S}_x^2 \rangle$ considered as a function of $|\langle \hat{S}_z \rangle|$ for states with N two mode bosons where the spin operators are chosen such that $\langle \hat{S}_x \rangle = \langle \hat{S}_y \rangle = 0$. Note that such spin operators can always be chosen so that the Bloch vector does lie along the z axis, even if the spin operators are not principal spin operators. Their treatment is based on combining the result from the Schwarz inequality

$$\langle \hat{S}_x^2 \rangle + \langle \hat{S}_y^2 \rangle + \langle \hat{S}_z \rangle^2 \leqslant J(J+1), \tag{113}$$

where $J = N/2$, and the Heisenberg uncertainty principle

$$\langle \Delta\hat{S}_x^2 \rangle \langle \Delta\hat{S}_y^2 \rangle = \xi \frac{1}{4} |\langle \hat{S}_z \rangle|^2, \tag{114}$$

where $\xi \geqslant 1$. In fact two inequalities can be obtained

$$\langle \Delta\hat{S}_x^2 \rangle \geqslant \frac{1}{2}\{(J(J+1) - \langle \hat{S}_z \rangle^2) - \sqrt{(J(J+1) - \langle \hat{S}_z \rangle^2)^2 - \xi \langle \hat{S}_z \rangle^2}\}, \tag{115}$$

$$\langle\Delta\widehat{S}_x^2\rangle \leqslant \frac{1}{2}\{(J(J+1) - \langle\widehat{S}_z\rangle^2) + \sqrt{(J(J+1) - \langle\widehat{S}_z\rangle^2)^2 - \xi\langle\widehat{S}_z\rangle^2}\} \quad (116)$$

which restricts the region in a $\langle\Delta\widehat{S}_x^2\rangle$ versus $|\langle\widehat{S}_z\rangle|$ plane that applies for states that are consistent with the Heisenberg uncertainty principle. As the derivation of the Heisenberg principle inequalities is not obvious, this is set out in appendix J. Note that in the first inequality the minimum value for $\langle\Delta\widehat{S}_x^2\rangle$ occurs for $\xi = 1$, and in the second inequality the maximum value for $\langle\Delta\widehat{S}_x^2\rangle$ also occurs for $\xi = 1$—the minimum HUP case. The first of these two inequalities is given as equation (3) in [30]. For states in which $\widehat{S}_x$ is squeezed relative to $\widehat{S}_y$ the points in the $\langle\Delta\widehat{S}_x^2\rangle$ versus $|\langle\widehat{S}_z\rangle|$ plane must also satisfy

$$\langle\Delta\widehat{S}_x^2\rangle \leqslant \frac{1}{2}|\langle\widehat{S}_z\rangle|. \quad (117)$$

Note that as $\widehat{J}_z$ is a spin angular momentum component we always have $|\langle\widehat{S}_z\rangle| \leqslant J$, which places an overall restriction on $|\langle\widehat{S}_z\rangle|$. However, for $\xi > 1$ there are values of $|\langle\widehat{S}_z\rangle|$ which are excluded via the Heisenberg uncertainty principle, since the quantity $(J(J+1) - \langle\widehat{S}_z\rangle^2)^2 - \xi\langle\widehat{S}_z\rangle^2$ then becomes negative. The question is: Is it possible to find values for $\langle\Delta\widehat{S}_x^2\rangle$ and $|\langle\widehat{S}_z\rangle|$ in which all three inequalities are satisfied? The answer is yes. This question is best examined via numerical calculations in which the regions where each inequality is satisfied are shown, and the results are presented in appendix J. Inspection of the three figures (figures 4–6 in the supplementary data). shows that there are regions where all three inequalities are satisfied.

Sørensen and Mølmer [30] also determine the minimum for $\langle\Delta\widehat{S}_x^2\rangle = \langle\widehat{S}_x^2\rangle$ as a function of $|\langle\widehat{S}_z\rangle|$ for various choices of J, subject to the constraints $\langle\widehat{S}_x\rangle = \langle\widehat{S}_y\rangle = 0$. The results show again that there is a region in the $\langle\Delta\widehat{S}_x^2\rangle$ versus $|\langle\widehat{S}_z\rangle|$ plane which is compatible with spin squeezing.

So although these considerations show that the Heisenberg uncertainty principle does not rule out extreme spin squeezing, nothing is yet directly determined about whether the spin squeezed states are entangled states for *modes* $\widehat{a}$, $\widehat{b}$, where the $\widehat{S}_\alpha$ are given as in equation (1). The discussion in [30] regarding entanglement is also based on using a density operator for non-entangled states as in equation (105) which only applies to distinguishable particles (see section 4.4). Sørensen [30] also showed that for higher J the amount of squeezing attainable could be greater. This fact enables a conclusion to be drawn from the measured spin variance about the minimum number of particles that participate in the non-separable component of an entangled state [31].

5. Correlation tests for entanglement

In section 2.4 of the accompanying paper I it was shown that for separable states the inequality $|\langle\widehat{\Omega}_A \otimes \widehat{\Omega}_B^\dagger\rangle|^2 \leqslant \langle\widehat{\Omega}_A^\dagger\widehat{\Omega}_A \otimes \widehat{\Omega}_B^\dagger\widehat{\Omega}_B\rangle$ applies, so that if

$$|\langle\widehat{\Omega}_A \otimes \widehat{\Omega}_B^\dagger\rangle|^2 > \langle\widehat{\Omega}_A^\dagger\widehat{\Omega}_A \otimes \widehat{\Omega}_B^\dagger\widehat{\Omega}_B\rangle \quad (118)$$

then the state is entangled. This is a general *correlation* test.

As will be seen the correlation tests can be re-expressed in terms of spin operators when dealing with SSR compliant states.

5.1. Dalton et al 2014

5.1.1. Weak correlation test for local SSR compliant non-entangled states.

For a non-entangled state based on *SSR compliant* $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ for modes $\widehat{a}$ and $\widehat{b}$ where the SSR is satisfied we have with $\widehat{\Omega}_A = (\widehat{a})^m$ and $\widehat{\Omega}_B = (\widehat{b})^n$

$$\langle(\widehat{a})^m(\widehat{b}^\dagger)^n\rangle = \sum_R P_R\langle(\widehat{a})^m(\widehat{b}^\dagger)^n\rangle_R = \sum_R P_R\langle(\widehat{a})^m\rangle_R\langle(\widehat{b}^\dagger)^n\rangle_R = 0 \quad (119)$$

since from equations analogous to (16) $\langle(\widehat{a})^m\rangle_R = \langle(\widehat{b}^\dagger)^n\rangle_R = 0$. Hence for a SSR compliant non-entangled state as defined in the present paper the inequality becomes

$$0 \leqslant \langle(\widehat{a}^\dagger)^m(\widehat{a})^m(\widehat{b}^\dagger)^n(\widehat{b})^n\rangle \quad (120)$$

which is trivially true and applies for *any* state, entangled or not.

Since $\langle(\widehat{a})^m(\widehat{b}^\dagger)^n\rangle$ is zero for non-entangled states it follows that it is merely necessary to show that this quantity is non-zero to establish that the state is entangled. Hence an *entanglement test* [2] in the case of sub-systems consisting of single modes $\widehat{a}$ and $\widehat{b}$ becomes

$$|\langle(\widehat{a})^m(\widehat{b}^\dagger)^n\rangle|^2 > 0 \quad (121)$$

for a non-entangled state based on *SSR compliant* $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$. Note that for globally compliant states $\langle(\widehat{a})^m(\widehat{b}^\dagger)^n\rangle = 0$ unless $n = m$, so only that case is of interest. This is a useful *weak correlation* test for entanglement in terms of the definition of entanglement in the present paper. A related but different test is that of Hillery *et al* [22]—discussed in the next subsection.

For the case where $n = m = 1$ the weak correlation test is

$$|\langle(\widehat{a}\,\widehat{b}^\dagger)\rangle|^2 > 0 \quad (122)$$

which is equivalent to $\langle\widehat{S}_x\rangle \neq 0$ and/or $\langle\widehat{S}_y\rangle \neq 0$, the Bloch vector test.

5.2. Hillery et al 2006, 2009

5.2.1. Hillery strong correlation entanglement test.

In a later paper entitled 'detecting entanglement with non-Hermitian operators' Hillery *et al* [32] apply other inequalities for determining entanglement derived in the earlier paper [22] but now also to systems of massive identical bosons, while still retaining density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ that contain coherences between states with differing boson numbers. In particular, for a non-entangled state the following family of inequalities

—originally derived in [22], is invoked.

$$|\langle(\hat{a})^m(\hat{b}^\dagger)^n\rangle|^2 \leqslant \langle(\hat{a}^\dagger)^m(\hat{a})^m(\hat{b}^\dagger)^n(\hat{b})^n\rangle. \tag{123}$$

This is just a special case of (118) with $\hat{\Omega}_A = (\hat{a})^m$ and $\hat{\Omega}_B = (\hat{b})^n$. Thus if

$$|\langle(\hat{a})^m(\hat{b}^\dagger)^n\rangle|^2 > \langle(\hat{a}^\dagger)^m(\hat{a})^m(\hat{b}^\dagger)^n(\hat{b})^n\rangle \tag{124}$$

then the state is entangled. The *Hillery et al* [22] entanglement test (124) is a valid test for entanglement and is actually a *more stringent test* than merely showing that $|\langle(\hat{a})^m(\hat{b}^\dagger)^n\rangle|^2 > 0$, since the quantity $|\langle(\hat{a})^m(\hat{b}^\dagger)^n\rangle|^2$ is now required to be *larger*. In a paper by He *et al* [23] (see section 5.3) the Hillery *et al* [22] entanglement test $|\langle(\hat{a})^m(\hat{b}^\dagger)^n\rangle|^2 > \langle(\hat{a}^\dagger)^m(\hat{a})^m(\hat{b}^\dagger)^n(\hat{b})^n\rangle$ is applied for the case where A and B *each* consist of *one mode* localized in each well of a double well potential. This test while applicable could be replaced by the more easily satisfied test $|\langle(\hat{a})^m(\hat{b}^\dagger)^n\rangle|^2 > 0$ (see (121)). However, as will be seen below in section 5.3, the Hillery *et al* [22] entanglement criterion is needed if the sub-systems each consist of *pairs of modes*, as treated in [23, 25].

Note that if $n \neq m$ the left side is zero for states that are globally SSR compliant. In this case we can always substitute for two mode systems

$$(\hat{a}\,\hat{b}^\dagger)^n = (\hat{S}_x - \mathrm{i}\hat{S}_y)^n,$$
$$(\hat{a}^\dagger)^n(\hat{a})^n = P_n(\hat{a}^\dagger\hat{a}) = P_n\left(\frac{\hat{N}}{2} - \hat{S}_z\right),$$
$$(\hat{b}^\dagger)^n(\hat{b})^n = P_n(\hat{b}^\dagger\hat{b}) = P_n\left(\frac{\hat{N}}{2} + \hat{S}_z\right) \tag{125}$$

(where P_n is a polynomial of order n) to write both the Hillery and the weak correlation test in terms of spin operators.

A particular case for $n = m = 1$ is the test $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 > \langle\hat{n}_a\,\hat{n}_b\rangle$ for an entangled state. To put this result in context, for a general quantum state and any operator $\hat{\Omega}$ we have $\langle\hat{\Omega}^\dagger\rangle = \langle\hat{\Omega}\rangle^*$ and $\langle(\hat{\Omega}^\dagger - \langle\hat{\Omega}^\dagger\rangle)(\hat{\Omega} - \langle\hat{\Omega}\rangle)\rangle \geqslant 0$, hence leading to the Schwarz inequality $|\langle\hat{\Omega}\rangle|^2 = |\langle\hat{\Omega}^\dagger\rangle|^2 \leqslant \langle\hat{\Omega}^\dagger\,\hat{\Omega}\rangle$. Taking $\hat{\Omega} = \hat{a}\,\hat{b}^\dagger$ leads to the inequality $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 \leqslant \langle\hat{n}_a(\hat{n}_b + 1)\rangle$, while choosing $\hat{\Omega} = \hat{b}\,\hat{a}^\dagger$ leads to the inequality $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 \leqslant \langle(\hat{n}_a + 1)\hat{n}_b\rangle$ for *all* quantum states. In both cases the right side of the inequality is greater than $\langle\hat{n}_a\,\hat{n}_b\rangle$, so *if* it was found that $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 > \langle\hat{n}_a\,\hat{n}_b\rangle$ (though of course still $\leqslant \langle\hat{n}_a(\hat{n}_b + 1)\rangle$ and $\leqslant \langle(\hat{n}_a + 1)\hat{n}_b\rangle$) then it could be concluded that the state was entangled. However, as we will see the left side $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2$ actually works out to be zero if physical states for $\hat{\rho}_R^A$, $\hat{\rho}_R^B$ are involved in defining non-entangled states, so that for a non-entangled state defined as in the present paper the true inequality replacing $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 \leqslant \langle\hat{n}_a\,\hat{n}_b\rangle$ is just $0 \leqslant \langle\hat{n}_a\,\hat{n}_b\rangle$, which is trivially true for any quantum state.

For the case where $n = m = 1$ we can write the test (124) in terms of spin operators using $\hat{a}\,\hat{b}^\dagger = \hat{S}_x - \mathrm{i}\hat{S}_y$ as

$$\langle\hat{S}_x\rangle_\rho^2 + \langle\hat{S}_y\rangle_\rho^2 > \frac{1}{4}\langle\hat{N}^2\rangle_\rho - \langle\hat{S}_z^2\rangle_\rho \tag{126}$$

which when combined with the general result $\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = (\hat{N}/2)(\hat{N}/2 + 1)$ leads to the test

$$\langle\Delta\hat{S}_x^2\rangle_\rho + \langle\Delta\hat{S}_y^2\rangle_\rho < \frac{1}{2}\langle\hat{N}\rangle_\rho. \tag{127}$$

This is the same as the Hillery spin variance test (84), so the Hillery first order correlation test does not add a further test for demonstrating non-SSR compliant entanglement. The Hillery correlation test for $n = 2$ leads to complex conditions involving higher powers of spin operators.

5.2.2. Applications of correlation tests for entanglement. As an example of applying these tests consider the *mixed two mode coherent states* described in appendix K, whose density operator for the two mode $\hat{a}$, $\hat{b}$ system is given in equation (326). We can now examine the Hillery *et al* [32] entanglement test in equation (124) and the entanglement test in equation (121) for the case where $m = n = 1$. It is straight-forward to show that

$$|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 = |\alpha|^4,$$
$$\langle(\hat{a}^\dagger\hat{a})(\hat{b}^\dagger\hat{b})\rangle = |\alpha|^4 \tag{128}$$

so that $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 = \langle(\hat{a}^\dagger\hat{a})(\hat{b}^\dagger\hat{b})\rangle$. A non-entangled state defined in terms of the SSR requirement for the separate modes satisfies $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 = 0$, while for a non-entangled state in which the SSR requirement for separate modes is not specifically required merely satisfies $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 \leqslant \langle(\hat{a}^\dagger\hat{a})(\hat{b}^\dagger\hat{b})\rangle$. Hence the test for entanglement of modes A, B in the present paper $|\langle\hat{a}\,\hat{b}^\dagger\rangle|^2 > 0$ is satisfied, while the Hillery *et al* [32] test $|\langle\hat{a}\hat{b}^\dagger\rangle|^2 > \langle(\hat{a}^\dagger\hat{a})(\hat{b}^\dagger\hat{b})\rangle$ is not.

In terms of the definition of non-entangled states in the present paper, the mixture of two mode coherent states given in equation (326) is *not* a *separable* state, but is an entangled state. As discussed in paper 1 (see section 3.4.3) this is because a coherent state gives rise to a non-zero coherence ($\langle\hat{a}\rangle \neq 0$) and thus cannot represent a physical state for the SSR compliant states involving identical massive bosons (as in BECs). However, in terms of the definition of non-entangled states in other papers such as those of Hillery *et al* [22, 32] the mixture of two mode coherent states would be a *non-entangled* state. It is thus a useful state for providing an example of the different outcomes of definitions where the local SSR is applied or not.

A further example of applying correlation tests is provided by the *NOON state* defined in (59) where here we consider modes A, B. All matrix elements of the form $\langle(\hat{a}^\dagger)^m(\hat{a})^m(\hat{b}^\dagger)^n(\hat{b})^n\rangle$ are zero for all m, n because both terms contain one mode with zero bosons. Matrix elements of the form $\langle(\hat{a})^m(\hat{b}^\dagger)^n\rangle$ are all zero unless $m = n = N$ and in this case

$$\langle(\hat{a})^N(\hat{b}^\dagger)^N\rangle = \langle(\hat{S}_x - \mathrm{i}\hat{S}_y)^N\rangle \\ = \cos\theta\,\sin\theta\,\langle N,0|(\hat{S}_-)^N|0,N\rangle \\ = \cos\theta\,\sin\theta\,\sqrt{N}\sqrt{2N-2} \\ \times\sqrt{3N-6}\sqrt{4N-12}\,...\sqrt{N} \qquad (129)$$

which is non-zero in general. Hence $|\langle(\hat{a})^N(\hat{b}^\dagger)^N\rangle|^2 > 0$ as required for both the weak and strong correlation tests, confirming that the NOON state is *entangled*. Carrying out this entanglement test experimentally for large N would involve measuring expectation values of high powers of the spin operators $\hat{S}_x$ and $\hat{S}_y$, which is difficult at present.

5.3. He et al 2012

For the *four mode* system associated with a double well described in section 4.2 (see [23]), the inequalities derived by Hillery *et al* [32] (see section 5.2)

$$|\langle(\hat{a}_i)^m(\hat{b}_j^\dagger)^n\rangle|^2 \leqslant \langle(\hat{a}_i^\dagger)^m(\hat{a}_i)^m(\hat{b}_j^\dagger)^n(\hat{b}_j)^n\rangle \qquad (130)$$

that apply for two non-entangled sub-systems A and B can now be usefully applied, since in this case the quantities $\langle(\hat{a}_i)^m(\hat{b}_j^\dagger)^n\rangle$ are in general no longer zero for separable states. Thus there is an *entanglement test*for two sub-systems consisting of *pairs of modes*. If

$$|\langle(\hat{a}_i)^m(\hat{b}_j^\dagger)^n\rangle|^2 > \langle(\hat{a}_i^\dagger)^m(\hat{a}_i)^m(\hat{b}_j^\dagger)^n(\hat{b}_j)^n\rangle \\ \text{for any of } i,j = 1,2 \qquad (131)$$

then the quantum state for two sub-systems A and B—*each* consisting of *two modes* localized in each well—is entangled. Again only the case where $m = n$ is relevant for states that are global SSR compliant.

6. Quadrature tests for entanglement

In this section we discuss tests for two mode entanglement involving so called *quadrature* operators—*position* and *momentum* being particular examples of such operators. These tests are distinct from those involving *spin* operators or *correlation* tests—the latter have been shown to be closely related to spin operator tests. The issue of *measurement* of the quadrature variances involved in these tests for the case of two mode systems involving identical massive bosons will be briefly discussed in section 7. Again we have a situation where tests derived in which local particle number SSR compliance for separable states is ignored are still valid when it is taken into account. However, when local particle number SSR compliance for separable states is actually included new entanglement tests arise. The *two mode quadrature squeezing* test in (155) is an example, though this test is not very useful as it could be replaced by the *Bloch vector* test. The quadrature correlation test in (152) also applies and is equivalent to the *Bloch vector* test. However the non-existent *quadrature variance* test in (141) is an example where there is no generalization of the previous entanglement test (see (133)) that applied when the SSR were irrelevant.

6.1. Duan et al 2000

6.1.1. Two distinguishable particles. A further inequality aimed at providing a signature for entanglement is set out in the papers by Duan *et al* [33], Toth *et al* [34]. Duan *et al* [33] considered a general situation where the system consisted of two distinguishable sub-systems A and B, for which *position* and *momentum* Hermitian operators $\hat{x}_A, \hat{p}_A$ and $\hat{x}_B, \hat{p}_B$ were involved that satisfied the standard commutation rules $[\hat{x}_A, \hat{p}_A] = [\hat{x}_B, \hat{p}_B] = i$ in units where $\hbar = 1$. These sub-systems were quite general and could be two *distinguishable* quantum particles A and B, but other situations can also be treated. An inequality was obtained for a two sub-system non-entangled state involving the variances for the commuting observables $\hat{x}_A + \hat{x}_B$ and $\hat{p}_A - \hat{p}_B$

$$\langle\Delta(\hat{x}_A + \hat{x}_B)^2\rangle + \langle\Delta(\hat{p}_A - \hat{p}_B)^2\rangle \geqslant 2 \qquad (132)$$

which could be used to establish a *variance test* for entangled states of the A and B sub-systems, so that if

$$\langle\Delta(\hat{x}_A + \hat{x}_B)^2\rangle + \langle\Delta(\hat{p}_A - \hat{p}_B)^2\rangle < 2 \qquad (133)$$

then the sub-systems are entangled. For the case of distinguishable particles such states are possible—consider for example any simultaneous eigenstate of the commuting observables $\hat{x}_A + \hat{x}_B$ and $\hat{p}_A - \hat{p}_B$. For such a state $\langle\Delta(\hat{x}_A + \hat{x}_B)^2\rangle$ and $\langle\Delta(\hat{p}_A - \hat{p}_B)^2\rangle$ are both zero, so the simultaneous eigenstates are entangled states of *particles* A, B. For simplicity we only set out the case for which $a = 1$ in [33]. The proof given in [33] considered separable states of the general form as in equation (15) for two sub-systems but where $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ are possible states for sub-systems A, B. Consequently, a first quantization case involving *one particle states* could be involved, where SSRs were *not* relevant. As explained in the introduction, the two distinguishable quantum particles are each equivalent to a whole set of single particle states (momentum eigenstates, harmonic oscillator states, ..) that each quantum particle can occupy, and because both $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ represent states for one particle we have $[\hat{n}_A, \hat{\rho}_R^A] = [\hat{n}_B, \hat{\rho}_R^B] = 0$. Because $\hat{\rho}$ represent a state for the two particles $[\hat{n}_A + \hat{n}_B, \hat{\rho}] = 0$, the SSR are still true, though irrelevant in the case of distinguishable quantum particles A and B.

Another inequality that can be established is

$$\langle\Delta(\hat{x}_A - \hat{x}_B)^2\rangle + \langle\Delta(\hat{p}_A + \hat{p}_B)^2\rangle \geqslant 2 \qquad (134)$$

which could also be used to establish a variance test for entangled states of the A and B sub-systems, so that if

$$\langle\Delta(\hat{x}_A - \hat{x}_B)^2\rangle + \langle\Delta(\hat{p}_A + \hat{p}_B)^2\rangle < 2 \qquad (135)$$

then the sub-systems are entangled.

6.1.2. Two mode systems of identical bosons. However, we can also consider cases of systems of *identical* bosons with two *modes* A, B rather than two *distinguishable* quantum particles A and B. In this case both the sub-systems may involve arbitrary numbers of particles, so it is of interest to see what implications follow from the physical sub-system states $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ now being required to satisfy the local particle number SSR, and all quantum states $\hat{\rho}$ satisfying the global particle number SSR. It is well-known that in two mode boson systems *quadrature operators* can be defined via

$$\hat{x}_A = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p}_A = \frac{1}{\sqrt{2}\,i}(\hat{a} - \hat{a}^\dagger),$$
$$\hat{x}_B = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger), \qquad \hat{p}_B = \frac{1}{\sqrt{2}\,i}(\hat{b} - \hat{b}^\dagger) \qquad (136)$$

which have the same commutation rules as the position and momentum operators for distinguishable particles. Thus $[\hat{x}_A, \hat{p}_A] = [\hat{x}_B, \hat{p}_B] = i$ as for cases where A, B were distinguishable particles.

Since the proof of equation (132) in [33] did not involve invoking the SSR, then *if* the inequality in equation (133) is satisfied, then the state *would be* an entangled state for *modes* A, B as well as for distinguishable particles A, B. The situation would *then* be similar to that for the Hillery *et al* [22, 32] tests—the SSR compliant sub-system states are just a particular case of the set of all sub-system states. However, in regard to spin squeezing and correlation tests for entanglement, new tests were found when the SSR were explicitly considered and it is *possible* that this could occur here. This turns out not to be the case.

As we will see, the inequality (132) is replaced by an *equation* that is satisfied by *all* quantum states for two mode systems of identical bosons where the global particle number SSR applies. This equation is the same irrespective of whether the state is separable or entangled. To see this we evaluate $\langle \Delta(\hat{x}_A + \hat{x}_B)^2\rangle + \langle \Delta(\hat{p}_A - \hat{p}_B)^2\rangle$ for states that are global SSR compliant.

Firstly,

$$\langle (\hat{x}_A + \hat{x}_B)\rangle = \langle (\hat{p}_A - \hat{p}_B)\rangle = 0 \qquad (137)$$

since $\langle \hat{a}\rangle = \langle \hat{b}\rangle = \langle \hat{a}^\dagger\rangle = \langle \hat{b}^\dagger\rangle = 0$ for SSR compliant states.

Secondly,

$$\langle (\hat{x}_A + \hat{x}_B)^2\rangle = \frac{1}{2}\begin{pmatrix} \langle \hat{a}\,\hat{a}^\dagger\rangle + \langle \hat{a}\,\hat{b}^\dagger\rangle + \langle \hat{b}\,\hat{a}^\dagger\rangle + \langle \hat{b}\,\hat{b}^\dagger\rangle \\ + \langle \hat{a}^\dagger\,\hat{a}\rangle + \langle \hat{a}^\dagger\,\hat{b}\rangle + \langle \hat{b}^\dagger\,\hat{a}\rangle + \langle \hat{b}^\dagger\,\hat{b}\rangle \end{pmatrix}$$

using $\langle \hat{a}^2\rangle = \langle (\hat{a}^\dagger)^2\rangle = \langle \hat{b}^2\rangle = \langle (\hat{b}^\dagger)^2\rangle = \langle \hat{a}\,\hat{b}\rangle = \langle \hat{a}^\dagger\,\hat{b}^\dagger\rangle = 0$ for global SSR compliant states. Hence using the commutation rules, introducing the number operator $\hat{N}$ and the spin operator $\hat{S}_x$ and using (137) we find that

$$\begin{aligned}\langle \Delta(\hat{x}_A + \hat{x}_B)^2\rangle &= \langle (\hat{x}_A + \hat{x}_B)^2\rangle \\ &= 1 + \langle \hat{a}^\dagger\,\hat{a}\rangle + \langle \hat{b}^\dagger\,\hat{b}\rangle + \langle \hat{b}^\dagger\,\hat{a}\rangle + \langle \hat{a}^\dagger\,\hat{b}\rangle \\ &= 1 + \langle \hat{N}\rangle + 2\langle \hat{S}_x\rangle. \end{aligned} \qquad (138)$$

Similarly

$$\langle \Delta(\hat{p}_A - \hat{p}_B)^2\rangle = 1 + \langle \hat{N}\rangle - 2\langle \hat{S}_x\rangle. \qquad (139)$$

Thus we have for all globally SSR compliant states

$$\langle \Delta(\hat{x}_A + \hat{x}_B)^2\rangle + \langle \Delta(\hat{p}_A - \hat{p}_B)^2\rangle = 2 + 2\langle \hat{N}\rangle. \qquad (140)$$

Since $\langle \hat{N}\rangle \geqslant 0$ for all quantum states we see that the Duan *et al* inequality (132) for separable states is still satisfied, but because (140) applies for all states irrespective of whether or not they are separable, we see that there is *no* quadrature variance entanglement test of the form

$$\langle \Delta(\hat{x}_A + \hat{x}_B)^2\rangle + \langle \Delta(\hat{p}_A - \hat{p}_B)^2\rangle < 2 + 2\langle \hat{N}\rangle \qquad (141)$$

for the case of two mode systems of identical massive bosons. The situation is similar to the non-existent test $\langle \Delta\hat{S}_x^2\rangle + \langle \Delta\hat{S}_y^2\rangle < |\langle \hat{S}_z\rangle|$ in section 4.1.3. The situation contrasts that in section 4.3, where a test $\langle \Delta(\hat{S}_x^1 + \hat{S}_x^2)^2\rangle + \langle \Delta(\hat{S}_y^1 - \hat{S}_y^2)^2\rangle < |\langle \hat{S}_z\rangle|$ establishes entanglement between two sub-systems (1 and 2)—but in this case each consisting of two modes.

We can also show for all globally SSR compliant states that

$$\langle \Delta(\hat{x}_A - \hat{x}_B)^2\rangle = 1 + \langle \hat{N}\rangle - 2\langle \hat{S}_x\rangle, \qquad (142)$$
$$\langle \Delta(\hat{p}_A + \hat{p}_B)^2\rangle = 1 + \langle \hat{N}\rangle + 2\langle \hat{S}_x\rangle \qquad (143)$$

and hence

$$\langle \Delta(\hat{x}_A - \hat{x}_B)^2\rangle + \langle \Delta(\hat{p}_A + \hat{p}_B)^2\rangle = 2 + 2\langle \hat{N}\rangle \qquad (144)$$

but again no entanglement test results.

The universal result (140) for the quadrature variance sum may seem paradoxical in view of the operators $(\hat{x}_A + \hat{x}_B)$ and $(\hat{p}_A - \hat{p}_B)$ commuting. Mathematically, this would imply that they would then have a complete set of simultaneous eigenvectors $|X_{A,B}, P_{A,B}\rangle$ such that $(\hat{x}_A + \hat{x}_B)|X_{A,B}, P_{A,B}\rangle = X_{A,B}|X_{A,B}, P_{A,B}\rangle$ and $(\hat{p}_A - \hat{p}_B)|X_{A,B}, P_{A,B}\rangle = P_{A,B}|X_{A,B}, P_{A,B}\rangle$. For these eigenstates $\langle \Delta(\hat{x}_A + \hat{x}_B)^2\rangle = \langle \Delta(\hat{p}_A - \hat{p}_B)^2\rangle = 0$ which contradicts (140) for such states. However, no such eigenstates exist that are globally SSR compliant. For SSR compliant states $|X_{A,B}, P_{A,B}\rangle$ must be an eigenstate of $\hat{N}$ and for eigenvalue N we see that $(\hat{x}_A + \hat{x}_B)\,|X_{A,B}, P_{A,B}\rangle_N$ is a linear combination of eigenstates of $\hat{N}$ with eigenvalues $N \pm 1$. Hence $(\hat{x}_A + \hat{x}_B)\,|X_{A,B}, P_{A,B}\rangle_N \neq X_{A,B}\,|X_{A,B}, P_{A,B}\rangle_N$ so simultaneous eigenstates that are SSR compliant do not exist and there is no paradox. As pointed out above, this issue does not arise for the case of two distinguishable particles where the operators $\hat{x}_A$, $\hat{x}_B$, $\hat{p}_A$ and $\hat{p}_B$ are *not* related to mode annihilation and creation operators—as in the present case.

We can also derive *inequalities* for *separable* states involving $\hat{x}_A, \hat{p}_A$ and $\hat{x}_B, \hat{p}_B$ based on the approach in section 4.3. Starting with appendix F equation (276) we choose $\hat{\Omega}_A = \hat{x}_A$, $\hat{\Omega}_B = \hat{x}_B$, $\hat{\Lambda}_A = \hat{p}_A$ and $\hat{\Lambda}_B = \hat{p}_B$. Here $\hat{\Theta}_A = \hat{1}_A$ and $\hat{\Theta}_B = \hat{1}_B$ For *separable* states we have from (276)

$$\langle \Delta(\alpha\hat{x}_A + \beta\hat{x}_B)^2\rangle + \langle \Delta(\alpha\hat{p}_A - \beta\hat{p}_B)^2\rangle \geqslant \alpha^2 + \beta^2. \qquad (145)$$

With the choice of $\alpha^2 = \beta^2 = 1$ we then find the following inequalities for separable states

$$\begin{aligned} &\langle \Delta(\hat{x}_A + \hat{x}_B)^2 \rangle + \langle \Delta(\hat{p}_A - \hat{p}_B)^2 \rangle \geqslant 2, \\ &\langle \Delta(\hat{x}_A - \hat{x}_B)^2 \rangle + \langle \Delta(\hat{p}_A + \hat{p}_B)^2 \rangle \geqslant 2, \\ &\langle \Delta(\hat{x}_A + \hat{x}_B)^2 \rangle + \langle \Delta(\hat{p}_A + \hat{p}_B)^2 \rangle \geqslant 2, \\ &\langle \Delta(\hat{x}_A - \hat{x}_B)^2 \rangle + \langle \Delta(\hat{p}_A - \hat{p}_B)^2 \rangle \geqslant 2 \end{aligned} \quad (146)$$

depending on the choice of α and β. With $\alpha = \beta = 1$ the first result is obtained and is the same as in (132). This result is consistent with (140). However using (138), (143), (142) and (139) we have for *global SSR compliant* states—separable *and* non-separable that the left sides of the last set of inequalities are respectively (a) $2 + 2\langle\hat{N}\rangle$, (b) $2 + 2\langle\hat{N}\rangle$, (c) $2 + 2\langle\hat{N}\rangle + 4\langle\hat{S}_x\rangle$ and (d) $2 + 2\langle\hat{N}\rangle + 4\langle\hat{S}_x\rangle$. The implications for the first two equalities have been discussed above. In the case of the $(+, +)$ and $(-, -)$ cases, we note that for states with eigenvalue N for $\hat{N}$ the eigenvalues for $\hat{S}_x$ lie in the range $-N/2$ to $+N/2$ and hence $\langle\hat{N}\rangle \pm 2\langle\hat{S}_x\rangle$ is always $\geqslant 0$. Thus (146) will apply for both separable *and* entangled states. Hence for global SSR compliant states none of (146) lead to an entanglement test.

6.1.3. Non SSR compliant states. On the other hand if *neither* the sub-system *nor* the overall system states are *required* to be SSR compliant—though they may be— we find that for *separable* states

$$\begin{aligned} &\langle \Delta(\hat{x}_A \pm \hat{x}_B)^2 \rangle_\rho + \langle \Delta(\hat{p}_A \mp \hat{p}_B)^2 \rangle_\rho \geqslant 2 + 2\langle \hat{N} \rangle_\rho \\ &\quad + 2(\langle \hat{a}\, \hat{b} \rangle + \langle \hat{a}^\dagger\, \hat{b}^\dagger \rangle) - 2|\langle \hat{a} \rangle_\rho + \langle \hat{b}^\dagger \rangle_\rho|^2 \end{aligned} \quad (147)$$

so entanglement based on ignoring local particle number SSR in the separable states is now shown if

$$\begin{aligned} &\langle \Delta(\hat{x}_A \pm \hat{x}_B)^2 \rangle_\rho + \langle \Delta(\hat{p}_A \mp \hat{p}_B)^2 \rangle_\rho < 2 + 2\langle \hat{N} \rangle_\rho \\ &\quad + 2(\langle \hat{a}\, \hat{b} \rangle_\rho + \langle \hat{a}^\dagger\, \hat{b}^\dagger \rangle_\rho) - 2|\langle \hat{a} \rangle_\rho + \langle \hat{b}^\dagger \rangle_\rho|^2. \end{aligned} \quad (148)$$

However, even if local particle number SSR compliance is ignored for the sub-system states (as in [4]), global particle number SSR compliance is still required for the overall quantum state. This applies to both the separable states and to states that are being tested for entanglement. In this case the quantities $\langle\hat{a}\hat{b}\rangle_\rho$, $\langle\hat{a}^\dagger\,\hat{b}^\dagger\rangle_\rho$, $\langle\hat{a}\rangle_\rho$, $\langle\hat{a}^\dagger\rangle_\rho$, $\langle\hat{b}\rangle_\rho$ and $\langle\hat{b}^\dagger\rangle_\rho$ are all zero, so the entanglement test in (148) would become the same as the *hypothetical* entanglement test (141).

For the sceptic (see appendix C) who wishes to completely disregard the SSR (both locally and globally) and proposes to use tests based on quadrature variances such as (148) to establish entanglement, the challenge will be to find a way of measuring the allegedly non-zero quantities $\langle\hat{a}\,\hat{b}\rangle_\rho$, $\langle\hat{b}^\dagger\rangle_\rho$. This would require some sort of system with a well-defined *phase* reference. Such a measurement is not possible with the beam splitter interferometer discussed in this paper, and the lack of such a detector system would preclude establishing SSR neglected entanglement for systems of identical bosons. Essentially the same problem arises in testing whether states that are non-SSR compliant exist in single mode systems of massive bosons.

As mentioned previously, the result in equation (132) was established in [33] *without* requiring the sub-system states $\hat{\rho}_R^A$, $\hat{\rho}_R^B$ to be compliant with the local particle number SSR or the density operator $\hat{\rho}$ for the state being tested to comply with the global particle number SSR, as would be the case for physical sub-system and system states of identical bosons. However, in [33] it was pointed out that *two mode squeezed vacuum* states of the form $|\Phi\rangle = \exp(-r(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}))|0\rangle$ satisfy the entanglement test. However, such stand alone two-mode states are *not* allowed quantum states for massive identical boson systems. as they are not compliant with the global particle number SSR. To create states with correlated pairs of bosons in modes *a* and *b* processes such as the *dissociation* of a bosonic *molecular BEC* in a mode *M* into *pair* of bosonic atoms in modes *a* and *b* can indeed occur, but would involve interaction Hamiltonians such as $\hat{V} = \kappa(\hat{a}^\dagger\hat{b}^\dagger\hat{M} + \hat{M}^\dagger\hat{a}\hat{b})$. The state produced would be an entangled state of the atoms plus molecules which would be compliant with the global total quanta number SSR—taking into account the boson particle content of the molecule via $\hat{N} = 2\hat{n}_M + \hat{n}_a + \hat{n}_b$. It would not be a state of the form $|\Phi\rangle = \exp(-r(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}))|0\rangle$.

6.2. Reid 1989

Another test involves the *general quadrature operators* defined as in [35], for which those in (136) are special cases

$$\begin{aligned} \hat{X}_a^\theta &= \frac{1}{\sqrt{2}}(\hat{a}\exp(-\mathrm{i}\theta) + \hat{a}^\dagger\exp(+\mathrm{i}\theta)), \\ \hat{X}_b^\phi &= \frac{1}{\sqrt{2}}(\hat{b}\exp(-\mathrm{i}\phi) + \hat{b}^\dagger\exp(+\mathrm{i}\phi)). \end{aligned} \quad (149)$$

These operators are Hermitian. The conjugate operators are

$$\begin{aligned} \hat{P}_a^\theta &= \frac{1}{\sqrt{2}\,i}(\hat{a}\exp(-\mathrm{i}\theta) - \hat{a}^\dagger\exp(+\mathrm{i}\theta)) = \hat{X}_a^{\theta+\pi/2}, \\ \hat{P}_b^\phi &= \frac{1}{\sqrt{2}\,i}(\hat{b}\exp(-\mathrm{i}\phi) - \hat{b}^\dagger\exp(+\mathrm{i}\phi)) = \hat{X}_b^{\phi+\pi/2}, \end{aligned} \quad (150)$$

where $[\hat{X}_a^\theta, \hat{P}_a^\theta] = [\hat{X}_b^\phi, \hat{P}_b^\phi] = i$.

Noting that for any state we have $\langle(\hat{X}_a^\theta - \lambda\hat{X}_b^\phi)^2\rangle \geqslant 0$ for all real λ establishes the Cauchy inequality for all quantum states

$$C_{ab}^{\theta\phi} = \frac{|\langle\hat{X}_a^\theta\hat{X}_b^\phi\rangle|^2}{\langle(\hat{X}_a^\theta)^2\rangle\langle(\hat{X}_b^\phi)^2\rangle} \leqslant 1. \quad (151)$$

The quantity $C_{ab}^{\theta\phi}$ is a *correlation coefficient*. For SSR compliant separable states $\langle\hat{X}_a^\theta\hat{X}_b^\phi\rangle = \sum_R P_R\langle\hat{X}_a^\theta\rangle_R\langle\hat{X}_b^\phi\rangle_R = 0$, while for all globally SSR compliant states $\langle(\hat{X}_a^\theta)^2\rangle = \langle\hat{n}_a\rangle + \frac{1}{2} > \frac{1}{2}$ and $\langle(\hat{X}_b^\phi)^2\rangle = \langle\hat{n}_b\rangle + \frac{1}{2} > \frac{1}{2}$. Hence for SSR compliant separable states the correlation coefficient is zero. A *quadrature correlation* test for entanglement based on locally SSR compliant sub-system states is then

$$C_{ab}^{\theta\phi} \neq 0. \quad (152)$$

However, it is not difficult to show that for states that are globally SSR compliant

$$\langle \widehat{X}_a^\theta \widehat{X}_b^\phi \rangle = \langle \widehat{S}_x \rangle \cos(\theta - \phi) + \langle \widehat{S}_y \rangle \sin(\theta - \phi) \quad (153)$$

so that the entanglement test based on locally SSR compliant sub-system states is equivalent to finding one of $\langle \widehat{S}_x \rangle$ or $\langle \widehat{S}_y \rangle$ to be non-zero. This is the *same* as the previous *Bloch vector* test in equation (52) or the weak correlation test in equation (122).

6.3. Two mode quadrature squeezing

From equation (149) we can define *two mode quadrature operators* as

$$\begin{aligned}\widehat{X}_\theta(+) &= \frac{1}{\sqrt{2}}(\widehat{X}_a^\theta + \widehat{X}_b^\theta) = \frac{1}{2}(\widehat{a}\exp(-i\theta) \\ &\quad + \widehat{b}^\dagger \exp(+i\theta) + \widehat{a}^\dagger \exp(+i\theta) + \widehat{b}\exp(-i\theta)),\\ \widehat{P}_\theta(+) &= \frac{1}{\sqrt{2}}(\widehat{P}_a^\theta + \widehat{P}_b^\theta) = \frac{1}{2i}(\widehat{a}\exp(-i\theta) \\ &\quad - \widehat{b}^\dagger \exp(+i\theta) - \widehat{a}^\dagger \exp(+i\theta) + \widehat{b}\exp(-i\theta)) \\ &= \widehat{X}_{\theta+\pi/2}(+),\end{aligned} \quad (154)$$

where we have $[\widehat{X}_\theta(+), \widehat{P}_\theta(+)] = i$. Note that $\widehat{X}_0(+) = (\widehat{x}_A + \widehat{x}_B)/\sqrt{2}$ and $\widehat{P}_0(+) = (\widehat{p}_A + \widehat{p}_B)/\sqrt{2}$ unlike the operators considered in section 6.1.2. As we have seen there is no entanglement test for systems of identical bosons of the form $\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \rangle + \langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \rangle < 2 + 2\langle \widehat{N} \rangle$. The Heisenberg uncertainty principle gives $\langle \Delta\widehat{X}_\theta^2(+) \rangle \langle \Delta\widehat{P}_\theta^2(+) \rangle \geqslant 1/4$, so a state is squeezed in $\widehat{X}_\theta(+)$ if $\langle \Delta\widehat{X}_\theta^2(+) \rangle < 1/2$, and similarly for squeezing in $\widehat{P}_\theta(+)$.

We can show that for separable states both $\langle \Delta\widehat{X}_\theta^2(+) \rangle \geqslant 1/2$ and $\langle \Delta\widehat{P}_\theta^2(+) \rangle \geqslant 1/2$, so two mode quadrature squeezing in either $\widehat{X}_\theta(+)$ or $\widehat{P}_\theta(+)$ is a test for two mode entanglement. The proof is given in appendix L. Hence the *two mode quadrature squeezing* test. If

$$\langle \Delta\widehat{X}_\theta^2(+) \rangle < \frac{1}{2} \quad \text{or} \quad \langle \Delta\widehat{P}_\theta^2(+) \rangle < \frac{1}{2} \quad (155)$$

then the state is entangled. Obviously $\widehat{X}_\theta(+)$ and $\widehat{P}_\theta(+)$ cannot both be squeezed for the same state.

We can also define additional two mode quadrature operators as

$$\begin{aligned}\widehat{X}_\theta(-) &= \frac{1}{\sqrt{2}}(\widehat{X}_a^\theta - \widehat{X}_b^\theta) = \frac{1}{2}(\widehat{a}\exp(-i\theta) \\ &\quad - \widehat{b}^\dagger \exp(+i\theta) + \widehat{a}^\dagger \exp(+i\theta) - \widehat{b}\exp(-i\theta)),\\ \widehat{P}_\theta(-) &= \frac{1}{\sqrt{2}}(\widehat{P}_a^\theta - \widehat{P}_b^\theta) = \frac{1}{2i}(\widehat{a}\exp(-i\theta) \\ &\quad + \widehat{b}^\dagger \exp(+i\theta) - \widehat{a}^\dagger \exp(+i\theta) - \widehat{b}\exp(-i\theta)) \\ &= \widehat{X}_{\theta+\pi/2}(-),\end{aligned} \quad (156)$$

where we also have $[\widehat{X}_\theta(-), \widehat{P}_\theta(-)] = i$. Again $\langle \Delta\widehat{X}_\theta^2(-) \rangle \langle \Delta\widehat{P}_\theta^2(-) \rangle \geqslant 1/4$, so a state is squeezed in $\widehat{X}_\theta(-)$ if $\langle \Delta\widehat{X}_\theta^2(-) \rangle < 1/2$, and similarly for squeezing in $\widehat{P}_\theta(-)$.

We can show that for separable states both $\langle \Delta\widehat{X}_\theta^2(-) \rangle \geqslant 1/2$ and $\langle \Delta\widehat{P}_\theta^2(-) \rangle \geqslant 1/2$, so two mode quadrature squeezing in either $\widehat{X}_\theta(-)$ or $\widehat{P}_\theta(-)$ is a test for two mode entanglement. The proof is given in appendix L. Hence the two mode quadrature squeezing test. If

$$\langle \Delta\widehat{X}_\theta^2(-) \rangle < \frac{1}{2} \quad \text{or} \quad \langle \Delta\widehat{P}_\theta^2(-) \rangle < \frac{1}{2} \quad (157)$$

then the state is entangled. Hence any one of $\widehat{X}_\theta(+)$, $\widehat{P}_\theta(+)$, $\widehat{X}_\theta(-)$, $\widehat{P}_\theta(-)$ being squeezed will demonstrate two mode entanglement.

The question then arises—Can two of the four two mode quadrature operators be squeezed? It turns out in general that only one of $\widehat{X}_\theta(+)$, $\widehat{P}_\theta(+)$, $\widehat{X}_\theta(-)$, $\widehat{P}_\theta(-)$ can be squeezed. This is shown in appendix L.

Further questions are: What quantities need to be measured in order to test whether two mode quadrature squeezing occurs and how useful would it be to detect entanglement? It is straight-forward to show from (154) and (156) that for states that are global SSR compliant

$$\langle \widehat{X}_\theta(+) \rangle = 0 \qquad \langle \widehat{X}_\theta(-) \rangle = 0, \quad (158)$$

$$\begin{aligned}\langle \Delta\widehat{X}_\theta^2(+) \rangle &= \langle \widehat{X}_\theta^2(+) \rangle = \frac{1}{2}(\langle \widehat{N} \rangle + 2\langle \widehat{S}_x \rangle),\\ \langle \Delta\widehat{X}_\theta^2(-) \rangle &= \langle \widehat{X}_\theta^2(-) \rangle = \frac{1}{2}(\langle \widehat{N} \rangle - 2\langle \widehat{S}_x \rangle)\end{aligned} \quad (159)$$

since terms such as $\langle \widehat{a}^2 \rangle$, $\langle \widehat{a}\widehat{b} \rangle$ etc are all zero for SSR compliant states. As explained in section D.4, both $\langle \widehat{N} \rangle + 2\langle \widehat{S}_x \rangle$ and $\langle \widehat{N} \rangle - 2\langle \widehat{S}_x \rangle$ are always non-negative, but the entanglement test would require

$$\begin{aligned}\langle \widehat{N} \rangle + 2\langle \widehat{S}_x \rangle &< 1 \quad \text{for squeezing in } \widehat{X}_\theta(+),\\ \langle \widehat{N} \rangle - 2\langle \widehat{S}_x \rangle &< 1 \quad \text{for squeezing in } \widehat{X}_\theta(-).\end{aligned} \quad (160)$$

This shows that the two mode quadrature squeezing test involves measuring $\langle \widehat{N} \rangle$ and $\langle \widehat{S}_x \rangle$, so that once again measurements of boson number and the mean value of a spin operator are involved. Similar conclusions apply for $\widehat{P}_\theta(+)$ and $\widehat{P}_\theta(-)$. However, since the test requires $\langle \widehat{S}_x \rangle$ to be non-zero it would simpler to use the *Bloch vector* test (see (52)) which merely requires showing that one of $\langle \widehat{S}_x \rangle$ or $\langle \widehat{S}_y \rangle$ to be non-zero.

In most cases the inequalities in (160) will not be satisfied, since both $\langle \widehat{N} \rangle$ and $\langle \widehat{S}_x \rangle$ are $O(N)$. However, for the binomial state in (61) with $\theta = 3\pi/4$ and $\chi = 0$ we have for $|\Phi\rangle = \left(\frac{-\widehat{a}^\dagger + \widehat{b}^\dagger}{\sqrt{2}}\right)^N |0\rangle/\sqrt{N!}$ the results $\langle \widehat{N} \rangle = N$ and $\langle \widehat{S}_x \rangle = -N/2$ (see (163) in [7]). Hence spin squeezing in $\widehat{X}_\theta(+)$ occurs, confirming that this particular binomial state is entangled. Note that the test does not confirm entanglement for almost all other binomial states (those where $\langle \widehat{S}_x \rangle$ is different from $\pm N/2$), though these are actually entangled.

7. Interferometry in bosonic systems

In this section we discuss how interferometers in two mode bosonic systems operate. This topic has of course been discussed many times before, but for completeness we present it here. Our approach is essentially the same as in earlier papers, for example that of Yurke *et al* [36]. Before discussing interferometry in two mode bosonic systems, we first need to set out the general Hamiltonian for the two mode systems that could be of interest. The two modes may be associated with two distinct single boson *spatial* states, such as in a double well potential in which case the coupling between the two modes is associated with *quantum tunneling*. Or they may be associated with two different atomic internal *hyperfine* states in a single well, which may be coupled via *classical fields* in the form of very short pulses, for which the time dependent amplitude is $\mathcal{A}(t)$, the center frequency is ω_0 and the *phase* is ϕ. Since this coupling process is much easier to control than quantum tunneling, we will mainly focus on the case of two modes associated with different hyperfine states, though the approach might also be applied to the case of two spatial modes. The free atoms occupying the two modes are associated with energies $\hbar\omega_a$, $\hbar\omega_b$, the *transition frequency* $\omega_{ba} = \omega_b - \omega_a$ being close to *resonance* with ω_0. It is envisaged that a large *number N* of *bosonic atoms* occupy the two modes. The bosonic atoms may also interact with each other via *spin conserving*, *zero range* interatomic potentials. We will show that measurements on the mean and variance for the *population difference* determine the *mean values* and *covariance matrix* for the spin operators involved in *entanglement* tests.

For interferometry involving *multi-mode* systems, a straightforward generalization of the two mode case is possible, based on the reasonable assumption the interferometer process couples the modes in a pair-wise manner. This is based on the operation of *selection rules*, as will be explained below.

However, although in the present section we show that two mode interferometers can be used to measure the *mean values* and *covariance matrix* for the *spin* operators involved in *entanglement* tests for systems of *massive* bosons, the issue of how to measure *mean values* and *variances* for the *quadrature* operators involved in other entanglement tests for massive bosons is still to be established. Such a measurement is not possible with the beam splitter interferometer discussed in this paper. An interferometer involving some sort of *phase reference* would seem to be needed. Proposals based on *homodyne* measurements have been made by Olsen *et al* [37, 38], but these are based on hypothetical reference systems with large boson numbers in *Glauber coherent* states, and such states are prohibited via the global particle number SSR.

7.1. Simple two mode interferometer

A simple description of the two mode system is provided by the *Josephson model*, where the overall Hamiltonian is of the form [7]

$$\widehat{H}_{\text{Joseph}} = \widehat{H}_0 + \widehat{V} + \widehat{V}_{\text{col}} \tag{161}$$

with

$$\begin{aligned}\widehat{H}_0 &= \hbar\omega_a\, \hat{a}^\dagger\hat{a} + \hbar\omega_b\, \hat{b}^\dagger\hat{b},\\ \widehat{V} &= \mathcal{A}(t)\exp(-\mathrm{i}\omega_0 t)\exp(\mathrm{i}\phi)\hat{b}^\dagger\hat{a}\\ &\quad + \mathcal{A}(t)\exp(+\mathrm{i}\omega_0 t)\exp(-\mathrm{i}\phi)\hat{a}^\dagger\hat{b},\\ \widehat{V}_{\text{col}} &= \chi(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a})^2,\end{aligned} \tag{162}$$

where $\widehat{H}_0$ is the free boson Hamiltonian, $\widehat{V}$ gives the interaction with the classical field and $\widehat{V}_{\text{col}}$ is the collisional interaction term. For the case of quantum tunneling between two distinct *spatial* modes, the interaction term $\widehat{V}$ can also be described in the Josephson model (see [7] for details), in which case the factors multiplying $\hat{b}^\dagger\hat{a}$ or $\hat{a}^\dagger\hat{b}$ involve the *trapping potential* and the two spatial mode functions. A time dependent amplitude and phase might be obtained via adding a suitable time dependent field to the trapping potential—this would be experimentally difficult. The Hamiltonian can also be written in terms of spin operators as

$$\begin{aligned}\widehat{H}_0 &= 1/2(\hbar\omega_a + \hbar\omega_b)\widehat{N} - \hbar\omega_{ab}\widehat{S}_z,\\ \widehat{V} &= \mathcal{A}(t)\exp(-\mathrm{i}\omega_0 t)\exp(\mathrm{i}\phi)(\widehat{S}_x + \mathrm{i}\widehat{S}_y) + \text{h.c.},\\ \widehat{V}_{\text{col}} &= 4\chi\widehat{S}_z^2.\end{aligned} \tag{163}$$

The coupling effect in a *simple two mode interferometer* can be described via the classical interaction term $\widehat{V}$, where now the amplitude $\mathcal{A}(t)$ is only non-zero over a short time interval. The pulsed classical field is associated with an *area variable s*, defined by

$$s = \int_{t_0}^{t} \mathrm{d}t_1 \mathcal{A}(t_1)/\hbar \tag{164}$$

the integral eventually being over the pulse's duration. The *interferometer frequency* ω_0 is assumed for simplicity to be in *resonance* with the *transition frequency* $\omega_{ba} = \omega_b - \omega_a$. The classical field is also associated with a phase ϕ, so the simple two mode interferometer is described by two *interferometric variables* $2s = \theta$ giving the pulse area and ϕ specifying the phase. Changing these variables leads to a range of differing applications of the interferometer. When acting as a *beam splitter* (BS) a $2s = \pi/2$ pulse is involved and ϕ is variable, but for a *phase changer* a $2s = \pi$ pulse is involved (ϕ is arbitrary). For *state tomography* in the *yz* plane we choose $2s = \theta$ (variable) and $\phi = 0$ or π. The beam splitter enables state tomography in the *xy* plane to be carried out. Generally speaking the effect of *collisions* can be *ignored* during the short classical pulse and we will do so here.

7.2. General two mode interferometers

More complex two mode bosonic interferometers applied to a specific input quantum state will involve specific *arrangements* of simple two mode interferometers such as beam splitters, phase changers and free evolution intervals, followed by final measurement of the mean population difference between the modes and its variance. *Ramsey interferometry* involves *two* beam splitters separated by a controllable free evolution *time interval T*. During such an interval in which free evolution occurs, the interaction of the classical beam splitter field with the two mode system can be ignored, but the effect of collisions and coupling to external systems may be important if collision parameters are to be measured using the interferometer. The overall behavior of such multi-element interferometers will also depend on the initial two mode quantum *input* state that acts as the *input state* for the interferometer, as well as important variables such as the phase ϕ, the center frequency ω_0, the area variable s for the classical pulses used, and also the the free evolution intervals (if any). The behavior also will depend on the characteristic parameters such as the transition frequency ω_{ab}, collision parameter χ and total boson number N for the two mode system used in the interferometry. The variables that describe the interaction with other systems whose properties are to be measured using the interferometer must also affect its behavior if the interferometer is to be useful. Finally, a choice must be made for the interferometer physical quantity whose *mean value* and *quantum fluctuation* is to be measured—referred to as the *measurable*. The outcome of such measurements can be studied as a function of one or more of the variables on which the interferometer behavior depends—referred to as the *interferometric variable*. There are obviously a wide range of possible two mode *interferometers types* that could be studied, depending on the application envisaged. Interferometers also have a wide range of *uses*, including determining the properties of the input two mode state—such as squeezing or entanglement. For a suitable known input state they can be used to measure interferometric variables—such as the classical phase ϕ of the pulsed field associated with a beam splitter or a parameter associated with an external system coupled to the interferometer. On the other hand, in a Ramsey interferometer the interferometric could be the collision parameter χ, obtainable if the free evolution period T is known. No attempt to be comprehensive will be made here.

The *Ramsey interferometer* is illustrated in figure 7.

For the purpose of considering entanglement tests a *simple two mode interferometer* operating under conditions of exact *resonance* $\omega_0 = \omega_{ab}$ will be treated, and its behavior for N large when the phase ϕ is changed and for different choices of the input state $\hat{\rho}$ will be examined. Measurements appropriate to detecting entanglement via *spin squeezing* and *correlation* will be discussed. The measurable chosen will initially be half the population difference $(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a})/2$—which equals $\hat{S}_z$—generally measured after the two mode system has interacted with the simple interferometer, but also without this interaction. The phase ϕ will act as the interferometric variable, as will the pulse area $2s = \theta$. As we will see, different choices of input state ranging from *separable* to *entangled* states lead to markedly different behaviors. In particular, the behavior of *relative phase eigenstates* as input states will be examined. Later we will also consider measurements involving the square of $\hat{S}_z$.

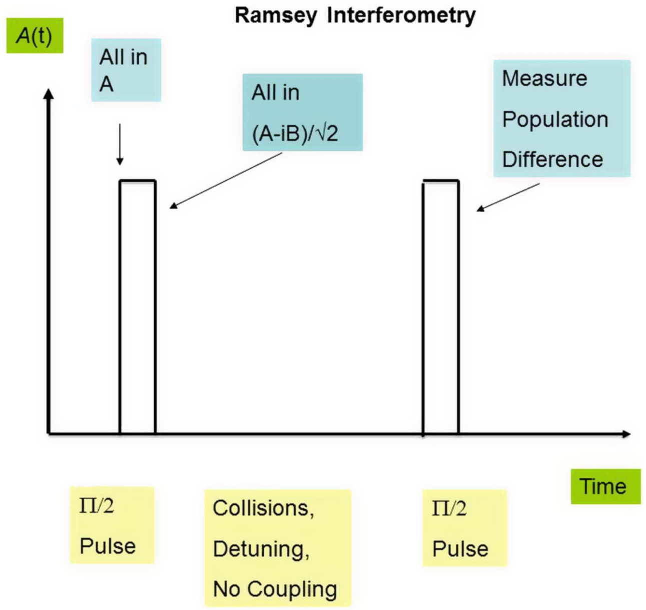

Figure 7. Ramsey interferometry. The pulse amplitude is shown as a function of time. For Ramsey interferometry there are two $\pi/2$ beam splitter pulses separated by a free evolution region, during which colisions may also occur. At the beginning of the first pulse all bosons are in mode A, at its end the bosons are in mode $(A - iB)/\sqrt{2}$. After the second pulse the population difference between modes A and B is measured.

7.3. Measurements in simple two mode interferometer

As discussed in the previous paragraph, the initial choice of *measurable* is

$$\widehat{M} = \frac{1}{2}(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a}) = \hat{S}_z \tag{165}$$

and we will determine its mean and variance for the state $\hat{\rho}^\#$ given by

$$\hat{\rho}^\# = \widehat{U}\,\hat{\rho}\,\widehat{U}^{-1}, \tag{166}$$

where the *output state* $\hat{\rho}^\#$ has evolved from the initial *input state* $\hat{\rho}$ due to the effect of the *simple two mode interferometer*. $\widehat{U}$ is the unitary *evolution operator* describing evolution during the time the short classical pulse is applied. Collision terms and interactions with other systems will be ignored during the short time interval involved.

We note that for an N boson state the eigenvalues of $\widehat{M}$ range from $-N/2$ to $+N/2$ in integer steps. For more general states the possible values for $\widehat{M}$ are any integer or half integer. When $\widehat{M}$ is measured the result will be one of these eigenvalues, but the average of repeated measurements will be $\langle\widehat{M}\rangle$ which must also lie in the range $-N/2$ to $+N/2$. The variance of the results for the repeated measurements of $\widehat{M}$ is also experimentally measurable and will not exceed $(N/2)^2$, and apart from *NOON* states will be much less than this. The experimentally determinable results for both $\langle\widehat{M}\rangle$ and $\langle\Delta\widehat{M}^2\rangle$ will depend on $\hat{\rho}$ and on the interferometer variables such as the phase ϕ and the pulse area $2s = \theta$.

The Hamiltonian governing the evolution in the simple two mode interferometer will be $\widehat{H}_0 + \widehat{V}$. For the *output state* the mean value and variance are

$$\langle \widehat{M} \rangle = \mathrm{Tr}(\widehat{M}\ \widehat{\rho}^{\#}),$$
$$\langle \Delta\widehat{M}^2 \rangle = \mathrm{Tr}(\{\widehat{M} - \langle \widehat{M} \rangle\}^2\ \widehat{\rho}^{\#}). \tag{167}$$

These will be evaluated at the end of the pulse. If the input state is measured *directly* without applying the interferometer, then the mean value and variance are as in the last equations but with $\widehat{\rho}^{\#}$ replaced by $\widehat{\rho}$.

The derivation of the results is set out in appendix M and are given by the same form as (167), but with $\widehat{\rho}^{\#}$ replaced by $\widehat{\rho}$ and with $\widehat{M}$ replaced by the interaction picture Heisenberg operator $\widehat{M}_H(2s,\ \phi)$ at the end of the pulse, which is given by

$$\widehat{M}_H(2s,\ \phi) = \frac{1}{2}(\widehat{b}_H^{\dagger}(s,\ \phi)\widehat{b}_H(s,\ \phi) - \widehat{a}_H^{\dagger}(s,\ \phi)\widehat{a}_H(s,\ \phi))$$
$$= \sin 2s(\sin\phi\ \widehat{S}_x + \cos\phi\ \widehat{S}_y) + \cos 2s\ \widehat{S}_z \tag{168}$$

with

$$\widehat{b}_H(s,\ \phi) = \cos s\ \widehat{b} - \mathrm{i}\exp(\mathrm{i}\phi)\sin s\ \widehat{a},$$
$$\widehat{a}_H(s,\ \phi) = -\mathrm{i}\exp(-\mathrm{i}\phi)\sin s\ \widehat{b} + \cos s\ \widehat{a}. \tag{169}$$

The versatility of the measurement follows from the range of possible choices of the pulse area $2s = \theta$ and the phase ϕ. These results are valid for both *bosonic* and *fermionic* modes.

We then find that the *general result* for the *mean* value is

$$\langle \widehat{M} \rangle = \sin\theta\ \sin\phi\ \langle \widehat{S}_x \rangle_\rho + \sin\theta\ \cos\phi\ \langle \widehat{S}_y \rangle_\rho + \cos\theta\ \langle \widehat{S}_z \rangle_\rho \tag{170}$$

and for the *variance* is

$$\begin{aligned}\langle \Delta\widehat{M}^2 \rangle =\ & \frac{(1 - \cos 2\theta)}{2}\frac{(1 - \cos 2\phi)}{2}C(\widehat{S}_x,\ \widehat{S}_x)\\ & + \frac{(1 - \cos 2\theta)}{2}\frac{(1 + \cos 2\phi)}{2}C(\widehat{S}_y,\ \widehat{S}_y)\\ & + \frac{(1 + \cos 2\theta)}{2}C(\widehat{S}_z,\ \widehat{S}_z)\\ & + \frac{(1 - \cos 2\theta)}{2}\sin 2\phi\ C(\widehat{S}_x,\ \widehat{S}_y)\\ & + \sin 2\theta\ \cos\phi\ C(\widehat{S}_y,\ \widehat{S}_z) + \sin 2\theta\ \sin\phi\ C(\widehat{S}_z,\ \widehat{S}_x),\end{aligned} \tag{171}$$

where the *mean value* of the spin operators are $\langle \widehat{S}_\alpha \rangle_\rho = \mathrm{Tr}(\widehat{S}_\alpha\ \widehat{\rho})$ and the *covariance matrix* elements are given by $C(\widehat{S}_\alpha,\ \widehat{S}_\beta) = 1/2\langle(\widehat{S}_\alpha\widehat{S}_\beta + \widehat{S}_\beta\widehat{S}_\alpha)\rangle_\rho - \langle \widehat{S}_\alpha \rangle_\rho \langle \widehat{S}_\beta \rangle_\rho$. The diagonal elements $C(\widehat{S}_\alpha,\ \widehat{S}_\alpha) = \langle \widehat{S}_\alpha{}^2 \rangle_\rho - \langle \widehat{S}_\alpha \rangle_\rho^2 = \langle \Delta\widehat{S}_\alpha^2 \rangle$ is the variance. By making appropriate choices of the interferometer variables θ (half the the pulse area) and ϕ (the phase) the mean values of all the spin operators and all elements of the covariance matrix can be measured. *Tomography* for the spin operators in any selected plane can be carried out.

An analogous treatment can be provided for the case of *multimode interferometers*. For the multi-mode case we consider two sets of modes $\widehat{a}_i$ and $\widehat{b}_i$ as described in appendix A. These may be different modes associated with two different hyperfine states or they may be modes associated with two separated potential wells. In addition, we assume the interferometer is based on *selection rules* which lead to pairwise coupling $\widehat{a}_i \leftrightarrow \widehat{b}_i$ between the modes. The theory gives the same results as in equations (170) and (171) for measurements of the mean value and variance of the half the population difference $\widehat{M}$ between the two sets of modes. The treatment is outlined in section M.5 of appendix M.

7.3.1. Tomography in xy plane—beam splitter. In the *beam splitter case* (for state tomography in the *xy* plane) we choose $2s = \pi/2$ and ϕ (variable) giving

$$\widehat{M}_H\left(\frac{\pi}{2},\ \phi\right) = \sin\phi\ \widehat{S}_x + \cos\phi\widehat{S}_y \tag{172}$$

and we find that for the output state of the *BS interferometer* the mean value and variance of $\widehat{M}$ are given by

$$\langle \widehat{M} \rangle = \sin\phi\ \langle \widehat{S}_x \rangle_\rho + \cos\phi\ \langle \widehat{S}_y \rangle_\rho, \tag{173}$$

$$\begin{aligned}\langle \Delta\widehat{M}^2 \rangle =\ & \frac{1}{2}(1 - \cos 2\phi)C(\widehat{S}_x,\ \widehat{S}_x)\\ & + \frac{1}{2}(1 + \cos 2\phi)C(\widehat{S}_y,\ \widehat{S}_y) + \sin 2\phi\ C(\widehat{S}_x,\ \widehat{S}_y)\end{aligned} \tag{174}$$

showing the mean value for the measurable $\widehat{M}$ depends sinusoidally on the phase ϕ and the *mean values* of the spin operators $\widehat{S}_x$, $\widehat{S}_y$. The variance for the measurable depends sinusoidally on 2ϕ and on the *covariance matrix* of the same spin operators. Both the means and covariances of the spin operators $\widehat{S}_x$, $\widehat{S}_y$ now depend on the input state $\widehat{\rho}$ for the interferometer rather than the output state $\widehat{\rho}^{\#}$.

7.3.2. Tomography in yz plane. For state tomography in the *yz* plane we obtain the means and covariances of the spin operators $\widehat{S}_y$, $\widehat{S}_z$. To do this we choose $2s = \theta$ (variable) and $\phi = 0$ so that

$$\widehat{M}_H(\theta,\ 0) = \sin\theta\ \widehat{S}_y + \cos\theta\ \widehat{S}_z \tag{175}$$

and find that for the output state the mean value and variance of $\widehat{M}$ are given by

$$\langle \widehat{M} \rangle = \sin\theta\ \langle \widehat{S}_y \rangle_\rho + \cos\theta\ \langle \widehat{S}_z \rangle_\rho, \tag{176}$$

$$\begin{aligned}\langle \Delta\widehat{M}^2 \rangle =\ & \frac{1}{2}(1 - \cos 2\theta)C(\widehat{S}_y,\ \widehat{S}_y)\\ & + \frac{1}{2}(1 + \cos 2\theta)C(\widehat{S}_z,\ \widehat{S}_z) + \sin 2\theta\ C(\widehat{S}_y,\ \widehat{S}_z).\end{aligned} \tag{177}$$

A *single* measurement does not of course determine the mean value $\langle \widehat{M} \rangle$. An average over a large number of *independent repetitions* of the measurement is needed to accurately determine $\langle \widehat{M} \rangle$ which can then be compared to theoretical predictions. This is a well-known practical issue for the experimenter that we need not dwell on here. A brief account of the issues involved is included in appendix N.

7.3.3. Phase changer. In this case we choose $2s = \theta = \pi$ and ϕ (arbitrary) giving

$$\widehat{M}_H(\pi, \phi) = -\widehat{S}_z \tag{178}$$

and for the output state the mean value and variance of $\widehat{M}$ are given by

$$\langle \widehat{M} \rangle = -\langle \widehat{S}_z \rangle_\rho, \tag{179}$$

$$\langle \Delta \widehat{M}^2 \rangle = \langle \Delta \widehat{S}_z^2 \rangle \tag{180}$$

so the phase changer measures the negative of the population difference. Effectively the phase changer interchanges the modes $\widehat{a} \to \widehat{b}$ and $\widehat{b} \to \widehat{a}$ and this is its role rather than being directly involved in a measurement. Phase changers are often included in complex interferometers at the midpoint of free evolution regions to cancel out unwanted causes of phase change.

7.3.4. Other measurements in simple two mode interferometer. Another useful choice of measurable is the *square* of the population difference

$$\widehat{M}_2 = \left(\frac{1}{2}(\widehat{b}^\dagger \widehat{b} - \widehat{a}^\dagger \widehat{a}) \right)^2 = \widehat{S}_z^2. \tag{181}$$

For the beam splitter case with $2s = \pi/2$ and ϕ (variable), we can easily show (see appendix M) that the mean value of $\widehat{M}_2$ for the output state is given by

$$\langle \widehat{M}_2 \rangle = \sin^2\phi \langle (\widehat{S}_x)^2 \rangle + \cos^2\phi \langle (\widehat{S}_y)^2 \rangle + \sin\phi \cos\phi \langle (\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x) \rangle \tag{182}$$

showing that the mean for the new observable $\widehat{M}_2$ is a sinusoidal function of the BS interferometer variable ϕ with coefficients that depend on the means of $\widehat{S}_x^2$, $\widehat{S}_y^2$ and $\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x$.

Choosing special cases for the interferometer variable yields the following useful results

$$\langle \widehat{M}_2 \rangle_{\phi=0} = \langle (\widehat{S}_y)^2 \rangle_\rho, \qquad \langle \widehat{M}_2 \rangle_{\phi=\pi/2} = \langle (\widehat{S}_x)^2 \rangle_\rho,$$
$$\langle \widehat{M}_2 \rangle_{\phi=\pi/4} - \langle \widehat{M}_2 \rangle_{\phi=-\pi/4} = \langle (\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x) \rangle_\rho. \tag{183}$$

Hence all three quantities $\langle (\widehat{S}_x)^2 \rangle$, $\langle (\widehat{S}_y)^2 \rangle$ and $\langle (\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x) \rangle$ can be measured. We note that just measuring $\langle \widehat{M}_2 \rangle$ does not add to the results obtained by measuring the mean and variance of the original measurable $\widehat{M}$, since $\langle \widehat{M}_2 \rangle = \langle \Delta \widehat{M}^2 \rangle + \langle \widehat{M} \rangle^2$. The variance $\langle \Delta \widehat{M}_2^2 \rangle$ does of course depend on higher moments, for example with $\phi = 0$ $\langle \Delta \widehat{M}_2^2 \rangle = \langle \Delta (\widehat{S}_y^2)^2 \rangle$ and $\phi = \pi/2$ $\langle \Delta \widehat{M}_2^2 \rangle = \langle \Delta (\widehat{S}_x^2)^2 \rangle$, so these also could be measured.

7.4. Application to spin squeezing tests for entanglement

Unless stated otherwise, we now focus on spin squeezing tests for *SSR compliant entanglement* based on the beam splitter measurements (the simple two mode interferometer with $2s = \theta = \pi/2$). By choosing the phase $\phi = 0$ we see that $\langle \widehat{M} \rangle = \langle \widehat{S}_y \rangle_\rho$ and $\langle \Delta \widehat{M}^2 \rangle = C(\widehat{S}_y, \widehat{S}_y) = \langle \{ \widehat{S}_y - \langle \widehat{S}_y \rangle_\rho \}^2 \rangle_\rho$ giving the mean and variance for the spin operator $\widehat{S}_y$. By choosing the phase $\phi = \pi/2$ we see that $\langle \widehat{M} \rangle = \langle \widehat{S}_x \rangle_\rho$ and $\langle \Delta \widehat{M}^2 \rangle = C(\widehat{S}_x, \widehat{S}_x) = \langle \{ \widehat{S}_x - \langle \widehat{S}_x \rangle_\rho \}^2 \rangle_\rho$ giving the mean and variance for the spin operator $\widehat{S}_x$. If the measurement of $\langle \widehat{M} \rangle$ were carried out without the beam splitter being present then $\langle \widehat{M} \rangle = \langle \widehat{S}_z \rangle_\rho$. Combining all these results then enables us to see whether or not $\widehat{S}_x$ is squeezed with respect to $\widehat{S}_y$ or vice versa. This illustrates the *use* of the interferometer in seeing if a state $\widehat{\rho}$ is *squeezed*. Squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_y$ (or $\widehat{S}_x$) or vice versa also demonstrates entanglement and again the simple two mode interferometer with appropriate choices of θ and ϕ can be used to measure the means and variances of the relevant spin operators.

As the presence of spin squeezing shows that the state must be entangled [2] the use of the interferometer for squeezing tests is important. Note that we still need to consider whether fluctuations due to a finite measurement sample could mask the test. However, as spin squeezing has been demonstrated in two mode systems of bosonic atoms this approach to demonstrating entanglement is clearly useful.

7.4.1. Spin squeezing in $\widehat{S}_x$, $\widehat{S}_y$. To demonstrate spin squeezing in $\widehat{S}_x$ with respect to $\widehat{S}_y$ we need to measure the variances $\langle \Delta \widehat{S}_x{}^2 \rangle_\rho$ and $\langle \Delta \widehat{S}_y{}^2 \rangle_\rho$ and the mean $\langle \widehat{S}_z \rangle_\rho$ and show that

$$\langle \Delta \widehat{S}_x{}^2 \rangle_\rho < \frac{1}{2} |\langle \widehat{S}_z \rangle_\rho| \qquad \langle \Delta \widehat{S}_y{}^2 \rangle_\rho > \frac{1}{2} |\langle \widehat{S}_z \rangle_\rho| \tag{184}$$

As we have seen, the variances in $\widehat{S}_y$, $\widehat{S}_x$ are obtained by measuring the *fluctuation* in the measurable $\widehat{M}$ *after* applying the interferometer to the state $\widehat{\rho}$, with the interferometer phase set to $\phi = 0$ or $\phi = \pi/2$ for the two cases respectively. The mean $\langle \widehat{S}_z \rangle_\rho$ is obtained by a direct measurement of the measurable $\widehat{M}$ *without* applying the interferometer to the state $\widehat{\rho}$.

7.4.2. Spin squeezing in xy plane. As shown in section 3 (see [2]) squeezing in $\widehat{S}_x$ compared to $\widehat{S}_y$ or vice versa is sufficient to show that the state is entangled. However, as the interferometer measures the variance for the state $\widehat{\rho}$ in the quantity

$$\widehat{M}_H(\pi/2, \phi) = \sin\phi\, \widehat{S}_x + \cos\phi\, \widehat{S}_y = \widehat{S}_x^{\#}\left(\frac{3\pi}{2} + \phi \right) \tag{185}$$

corresponding to the x component of the spin vector operator ($\overrightarrow{\widehat{S}}$) after it has been rotated about the z axis through an angle $\frac{3\pi}{2} + \phi$, it is desirable to *extend* the entanglement test to consider the squeezing of $\widehat{S}_x^{\#}\left(\frac{3\pi}{2} + \phi \right)$ with respect to the corresponding y component $\widehat{S}_y^{\#}\left(\frac{3\pi}{2} + \phi \right)$—and vice versa, where

$$\widehat{S}_y^{\#}\left(\frac{3\pi}{2} + \phi \right) = -\cos\phi\, \widehat{S}_x + \sin\phi\, \widehat{S}_y. \tag{186}$$

The variance in $\widehat{S}_y^{\#}\left(\frac{3\pi}{2} + \phi \right)$ can be obtained by changing the interferometer phase to $\phi + \frac{\pi}{2}$. Clearly $\left[\widehat{S}_x^{\#}\left(\frac{3\pi}{2} + \phi\right), \widehat{S}_y^{\#}\left(\frac{3\pi}{2} + \phi\right)\right] = i\,\widehat{S}_z$, as before.

The question is—does squeezing in either $\widehat{S}_x^{\#}\left(\frac{3\pi}{2} + \phi\right)$ or $\widehat{S}_y^{\#}\left(\frac{3\pi}{2} + \phi\right)$ demonstrate entanglement of the modes $\widehat{a}$ and $\widehat{b}$? The answer is that it does. The proof is set out in section M.4 of appendix M.

7.4.3. Measurement of $\langle \widehat{S}_z \rangle_\rho$. The question remaining is whether the mean value $\langle \widehat{S}_z \rangle_\rho$ can be measured accurately enough to apply the test for entanglement. With an infinite number of repeated measurements this is always possible, since then both the variances $\left\langle \Delta \widehat{S}_x^{\#}\left(\frac{3\pi}{2} + \phi\right)^2 \right\rangle_\rho$ and the mean $\langle \widehat{S}_z \rangle_\rho$. would become error free. For a finite number of measurements R the measurement of $\langle \widehat{S}_z \rangle_\rho$ requires a consideration of the variance in $\widehat{S}_z$. For *general* entangled states general considerations indicate that this mean will be of order N and the variance will be at worst of order N^2, Hence the variance $\langle \Delta \langle \widehat{S}_z \rangle^2 \rangle_{\text{sample}}$ in determining the mean $\langle \widehat{S}_z \rangle$ for R repetitions of the measurement would be $\sim N^2/R$, giving a fluctuation of $\sim N/\sqrt{R}$. For this to be small compared to $\sim N$ we merely require $R \gg 1$, which is not unexpected. This result indicates that the application of the spin squeezing test via interferometric measurement of both the variances $\left\langle \Delta \widehat{S}_x^{\#}\left(\frac{3\pi}{2} + \phi\right)^2 \right\rangle_\rho$ and the mean $\langle \widehat{S}_z \rangle_\rho$ looks feasible.

7.4.4. Spin squeezing in $\widehat{S}_z$, $\widehat{S}_y$. To demonstrate spin squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_y$ we need to measure the variances $\langle \Delta \widehat{S}_z{}^2 \rangle_\rho$ and $\langle \Delta \widehat{S}_y{}^2 \rangle_\rho$ and the mean $\langle \widehat{S}_x \rangle_\rho$ and show that

$$\langle \Delta \widehat{S}_z{}^2 \rangle_\rho < \frac{1}{2} |\langle \widehat{S}_x \rangle_\rho|, \qquad \langle \Delta \widehat{S}_y{}^2 \rangle_\rho > \frac{1}{2} |\langle \widehat{S}_x \rangle_\rho|. \tag{187}$$

As we have seen, the variances in $\widehat{S}_z$, $\widehat{S}_y$ are obtained by measuring the *fluctuation* in the measurable $\widehat{M}$ *after* applying the interferometer to the state $\widehat{\rho}$, with the interferometer phase set to $\phi = 0$ and the pulse area $2s = \theta$ made variable. From equation (177) we see that choosing $\theta = 0$ gives $\langle \Delta \widehat{S}_z{}^2 \rangle_\rho$ and choosing $\theta = \frac{\pi}{2}$ gives $\langle \Delta \widehat{S}_y{}^2 \rangle_\rho$. From equation (173) the mean $\langle \widehat{S}_x \rangle_\rho$ is obtained by a measurement of the *mean* in the measurable $\widehat{M}$ *after* applying the interferometer to the state $\widehat{\rho}$, with the interferometer phase set to $\phi = \pi/2$ and the pulse area $2s = \pi/2$.

7.5. Application to correlation tests for entanglement

7.5.1. First order correlation test. Unless stated otherwise, we again focus on correlation tests for *SSR compliant entanglement*. For the beam splitter case and by choosing the phase $\phi = 0$ we see that $\langle \widehat{M} \rangle = \langle \widehat{S}_y \rangle_\rho$ and by choosing the phase $\phi = \pi/2$ we see that $\langle \widehat{M} \rangle = \langle \widehat{S}_x \rangle_\rho$. The simplest form of the correlation test with $n = m = 1$ requires

$$\langle \widehat{S}_x \rangle_\rho \neq 0, \qquad \langle \widehat{S}_y \rangle_\rho \neq 0 \tag{188}$$

for establishing that the state is entangled. For the separable state with $\widehat{M}_H = \sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y = \widehat{S}_x^{\#}\left(\frac{3\pi}{2} + \phi\right)$

$$\langle \widehat{M} \rangle = \langle \widehat{M}_H \rangle_\rho = 0 \tag{189}$$

so that the mean value of the measurable is zero and independent of the beam splitter phase ϕ for all ϕ. Finding any non-zero value for $\langle \widehat{M}_H \rangle_\rho$ would then show that the state $\widehat{\rho}$ is entangled. More importantly from the general result, $\langle \widehat{M}_H \rangle_\rho$ would show a *sinusoidal dependence* on the interferometer phase ϕ, so the appearance of such a dependence would be an indication that the state was entangled.

The question remaining is whether the mean values $\langle \widehat{S}_{x,y} \rangle_\rho$ can be measured accurately enough to apply the test for entanglement. With an infinite number of repeated measurements this is always possible, since then both the variances $\langle \Delta \widehat{S}_{x,y}^2 \rangle_\rho$ and the means $\langle \widehat{S}_{x,y} \rangle_\rho$. would become error free. For a finite number of measurements the measurement of $\langle \widehat{S}_{x,y} \rangle_\rho$ requires a consideration of the variances in $\widehat{S}_{x,y}$ (or $\widehat{M}_H$ to cover both cases). For a general entangled state we can assume that $\langle \widehat{M}_H \rangle_\rho \sim N/2$ and the variance will be at worst of order N^2, Hence the variance $\langle \Delta \langle \widehat{M}_H \rangle^2 \rangle_{\text{sample}}$ in determining the mean $\langle \widehat{S}_{x,y} \rangle$ for R repetitions of the measurement would be $\sim N^2/R$, giving a fluctuation of $\sim N/\sqrt{R}$. For this to be small compared to $\sim N$ we merely require $R \gg 1$, which is not unexpected. This result indicates that the application of the simple correlation test via interferometric measurement of $\langle \widehat{S}_x \rangle_\rho$ and $\langle \widehat{S}_y \rangle_\rho$ looks feasible.

7.5.2. Second order correlation test. For the second order form of the correlation test with $n = m = 2$ requires

$$\langle \Delta \widehat{S}_x{}^2 \rangle_\rho + \langle \widehat{S}_x \rangle_\rho^2 - \langle \Delta \widehat{S}_y{}^2 \rangle_\rho - \langle \widehat{S}_y \rangle_\rho^2 \neq 0$$
$$(C(\widehat{S}_x, \widehat{S}_y) + \langle \widehat{S}_x \rangle_\rho \langle \widehat{S}_y \rangle_\rho) \neq 0. \tag{190}$$

We have already shown using equations (173) and (174) that the variances $\langle \Delta \widehat{S}_y{}^2 \rangle_\rho$ and $\langle \Delta \widehat{S}_x{}^2 \rangle_\rho$ and the means $\langle \widehat{S}_y \rangle_\rho$ and $\langle \widehat{S}_x \rangle_\rho$ can be obtained via the BS interferometer from the mean $\langle \widehat{M} \rangle$ and the variance $\langle \Delta \widehat{M}^2 \rangle$ for the choices of $\phi = 0$ and $\phi = \pi/2$. To obtain the covariance matrix element $C(\widehat{S}_x, \widehat{S}_y)$ we see that if we choose $\phi = \pi/4$ then $\langle \Delta \widehat{M}^2 \rangle = \frac{1}{2} \langle \Delta \widehat{S}_x{}^2 \rangle_\rho + \frac{1}{2} \langle \Delta \widehat{S}_y{}^2 \rangle_\rho + C(\widehat{S}_x, \widehat{S}_y)$, from which the covariance can be measured. Thus the second order correlation test can be applied.

Alternately, if the measurement quantity for the BS interferometer is the square of the population difference then we see from (183) that the mean value of $\widehat{M}_2$ for certain choices of the BS variable ϕ measures $\langle \Delta \widehat{S}_x{}^2 \rangle_\rho + \langle \widehat{S}_x \rangle_\rho^2 = \langle \widehat{S}_x{}^2 \rangle_\rho$, $\langle \Delta \widehat{S}_y{}^2 \rangle_\rho + \langle \widehat{S}_y \rangle_\rho^2 = \langle \widehat{S}_y{}^2 \rangle_\rho$ and $(C(\widehat{S}_x, \widehat{S}_y) + \langle \widehat{S}_x \rangle_\rho \langle \widehat{S}_y \rangle_\rho) = \langle (\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x) \rangle_\rho$. The second order correlation test is that if

$$\langle \widehat{S}_x{}^2 \rangle_\rho \neq \langle \widehat{S}_y{}^2 \rangle_\rho$$
$$\langle (\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x) \rangle_\rho \neq 0 \tag{191}$$

then the state is entangled.

7.6. Application to quadrature tests for entanglement

As we saw previously, no useful quadrature test for SSR compliant entanglement in two mode systems of identical bosons of the form $\langle\Delta(\hat{x}_A \pm \hat{x}_B)^2\rangle_\rho + \langle\Delta(\hat{p}_A \mp \hat{p}_B)^2\rangle_\rho < 2 + 2\langle\hat{N}\rangle_\rho$ results if the overall system state is globally SSR compliant. However, if the *separable* states are non-compliant then showing that

$$\langle\Delta(\hat{x}_A \pm \hat{x}_B)^2\rangle_\rho + \langle\Delta(\hat{p}_A \mp \hat{p}_B)^2\rangle_\rho < 2 + 2\langle\hat{N}\rangle_\rho + 2(\langle\hat{a}\,\hat{b}\rangle_\rho + \langle\hat{a}^\dagger\,\hat{b}^\dagger\rangle_\rho) - 2|\langle\hat{a}\rangle_\rho + \langle\hat{b}^\dagger\rangle_\rho|^2. \qquad (192)$$

would demonstrate entanglement. This test requires measuring $\langle\hat{N}\rangle_\rho$ together with $\langle\hat{a}\,\hat{b}\rangle_\rho$, $\langle\hat{a}\rangle_\rho$ and $\langle\hat{b}^\dagger\rangle_\rho$. Although the first can be measured using the BS interferometer the quantities $\langle\hat{a}\,\hat{b}\rangle_\rho$, $\langle\hat{a}\rangle_\rho$ and $\langle\hat{b}^\dagger\rangle_\rho$ cannot. Another technique involving a measuring system where there is a well-defined *phase reference* is therefore required if quadrature tests for SSR neglected entanglement are to be undertaken. Furthermore, the *overall* state must still be globally SSR compliant, and hence $\langle\hat{a}\,\hat{b}\rangle_\rho$, $\langle\hat{a}\rangle_\rho$ and $\langle\hat{b}^\dagger\rangle_\rho$ are all zero, even for entangled states, so the test reduces to $\langle\Delta(\hat{x}_A \pm \hat{x}_B)^2\rangle_\rho + \langle\Delta(\hat{p}_A \mp \hat{p}_B)^2\rangle_\rho < 2 + 2\langle\hat{N}\rangle_\rho$. Since for *all* states $\langle\Delta(\hat{x}_A \pm \hat{x}_B)^2\rangle_\rho + \langle\Delta(\hat{p}_A \mp \hat{p}_B)^2\rangle_\rho = 2 + 2\langle\hat{N}\rangle_\rho$ this test must fail anyway. We have also seen that finding the *correlation coefficient*—defined in terms of generalized quadrature operators (149) in equation (151)—to be non-zero does not lead to a new test for SSR compliant entanglement. The tests involving *two mode quadrature squeezing* look more promising, assuming the relevant quadrature variances can be measured.

8. Experiments on spin squeezing

We now examine a number of recent experimental papers involving squeezing and entanglement in BEC with *large* numbers of *identical* bosons. Their notation will be modified to be the same as here. There are really two questions to consider. One is whether squeezing has been created (and of which type). The second is whether or not this demonstrates entanglement of the modes involved. Here we define entanglement for identical bosons as set out in section 3 of paper I. Many of these experiments involve Ramsey interferometers and the aim was to demonstrate spin squeezing created via the collisional interaction between the bosons. Obviously, in demonstrating *spin squeezing* they would hope to have created an entangled state, though in most cases an entangled state had already been created via the interaction with the first beam splitter. Although the criterion for entanglement used in most cases was based on an experimental *proposal* [14, 30] which regarded identical particles as *distinguishable* subsystems, the spin squeezing test based on $\hat{S}_z$ does turn out to be a valid test for two mode entanglement, as explained in section 3. However, it should be noted that all the papers discussed have a different viewpoint regarding what exactly is entangled—generally referring to entanglement of *atoms* or *particles* rather than modes. All the experiments discussed below establish entanglement, though often this was already created in a first $\pi/2$ coupling pulse. Most are based on the spin squeezing test involving $\hat{S}_z$, that of Gross *et al* [39] involved *population difference squeezing* rather than spin squeezing (see section 3.4). The other experiment of Gross *et al* [40] shows (see figure 2(b) in [40]) that the mean value of one of the two spin operators $\hat{S}_x$, $\hat{S}_y$ is non-zero, measurements based on (173) for the simple two mode interferometer with $2s = \pi/2$, ϕ (variable) showing that the Bloch vector lies on the Bloch sphere. This is sufficient to demonstrate two mode entanglement, as (52) shows.

A key result of the present paper (and [2]) is that the conclusion that experiments which have demonstrated spin squeezing in $\hat{S}_z$ have thereby demonstrated two mode entanglement, no longer has to be justified on the basis of a proof that clearly does not apply to a system of identical bosons.

8.1. Estève et al [41]

- Stated emphasis—generation of spin squeezed states suitable for atom interferometry, demonstration of *particle* entanglement.
- System—Rb^{87} in two hyperfine states.
- BEC of Rb^{87} trapped in optical lattice superposed on harmonic trap.
- Occupation number per site 100–1100 atoms—macroscopic.
- Situation where atoms trapped in just two sites treated—two mode entanglement?
- Claimed observed (see figure 1 in [41]) spin squeezing based on $N\langle\Delta\hat{S}_z^2\rangle/(\langle\hat{S}_x^2\rangle + \langle\hat{S}_y^2\rangle) < 1$ (see (107)).
- Claimed entanglement of identical atoms.
- Spin squeezing test is based on assumption that Bloch vector is on Bloch sphere, a result not established.
- Comment—spin squeezing in $\hat{S}_z$ (almost) demonstrated (see (107)), so entanglement is established.

8.2. Riedel et al [42]

- Stated emphasis—generation of spin squeezed states suitable for atom interferometry, demonstration of *multi-particle* entanglement.
- System—Rb^{87} in two hyperfine states.
- BEC of Rb^{87} trapped in harmonic trap with non-zero magnetic field—Zeeman splitting of levels.
- Number of atoms 1200—macroscopic.
- Process involves Ramsey interferometry—starts with all atoms in one state, $\pi/2$ pulse (duration $\pi/2\Omega$?) generates coherent spin state $(\hat{a}^\dagger + \hat{b}^\dagger)^N|0\rangle$ (entangled), free evolution with collisions (causes squeezing), second pulse with area $2s = \theta$ and phase π (or 0) followed by detection of population difference—associated with operator $\hat{S}_z$.

- Evolution described using Josephson Hamiltonian $\widehat{H} = \delta\widehat{S}_z + \Omega\widehat{S}_\phi + \chi\widehat{S}_z^2$ where $\widehat{S}_\phi = \cos\phi\,\widehat{S}_x - \sin\phi\,\widehat{S}_y$, Ω is Rabi frequency, ϕ is phase of RF-microwave field, δ is detuning, χ describes collisions. Interaction picture and on resonance?
- During free evolution including collisions spatial modes for internal states pushed apart so that χ becomes much bigger in order to give larger squeezing.
- Final pulse enables state tomography in the yz plane to be carried out—measures spin squeezing for spin operator $\widehat{S}_\theta = \cos\theta\,\widehat{S}_z - \sin\theta\,\widehat{S}_y$ in this plane (see appendix M (363) herein with $\phi = \pi$).
- Claimed observed spin squeezing based on $\langle\Delta\widehat{S}_\theta^2\rangle$ being less than standard quantum limit $N/4$. (see figure 2 in [42]).
- No measurement made to show that $|\langle\widehat{S}_x\rangle| \approx N/2$ as required to justify spin squeezing test. Spin squeezing test is based on assumption that Bloch vector is on Bloch sphere, a result not established.
- Claim that state of atoms at end of free evolution is four-partite entangled based on spin squeezing test is not substantiated, also an entangled state was already created by first $\pi/2$ pulse.
- Comment—spin squeezing in $\widehat{S}_z$ (almost) demonstrated (see (107)), so entanglement is established. An entangled state was of course already created by first $\pi/2$ pulse, and then modified via the collisional effects.

8.3. Gross et al [40]

- Stated emphasis—generation of non-classical spin squeezed states for nonlinear atom interferometry, demonstration of entanglement between *atoms*.
- System—Rb^{87} in two hyperfine states.
- Six independent BECs of Rb^{87} trapped in six separate wells in a optical lattice.
- Number of atoms 2300—macroscopic, down to ca 170 in each well.
- Evolution described using Josephson Hamiltonian $\widehat{H} = \Delta\omega_0\widehat{S}_z + \Omega\widehat{S}_\gamma + \chi\widehat{S}_z^2$ where (in the present notation) $\widehat{S}_\gamma = \cos\gamma\,\widehat{S}_x + \sin\gamma\,\widehat{S}_y$, Ω is Rabi frequency, γ is phase of RF-microwave field, $\Delta\omega_0$ is detuning, χ describes collisions. Interaction picture and on resonance ?
- During free evolution plus collision evolution Feshach resonance used so that χ becomes much bigger in order to give larger squeezing.
- One process involves Ramsey interferometry—starts with all atoms in one state, $\pi/2$ pulse (duration $\pi/2\Omega$?) generates coherent spin state $(\widehat{a}^\dagger + \widehat{b}^\dagger)^N|0\rangle$ (entangled) with $\langle\widehat{S}_z\rangle = 0$, free evolution with collisions (causes squeezing) and with spin echo pulse applied, second $\pi/2$ pulse with phase ϕ followed by detection of population difference—associated with operator $\widehat{S}_z$.
- Population difference measured after last $\pi/2$ pulse shows a sinusoidal dependence on phase ϕ (see figure 2(b) in [40]). This shows that $\langle\widehat{S}_x\rangle$ and $\langle\widehat{S}_y\rangle$ are non-zero, thereby showing that the state created just prior to last pulse is entangled (see Bloch vector test (52)). This does not of course show that the state is spin squeezed.
- Another process involves generation of coherent spin state $(\widehat{a}^\dagger + \widehat{b}^\dagger)^N|0\rangle$ (entangled) with $\langle\widehat{S}_z\rangle = 0$, then free evolution with collisions (causes squeezing), followed by coupling pulse to rotate Bloch vector through angle α thereby crossing xy plane. The variance in $\widehat{S}_z$ is then measured as α changes.
- Claimed observed spin squeezing based on $N\langle\Delta\widehat{S}_z^2\rangle/(\langle\widehat{S}_x\rangle^2 + \langle\widehat{S}_y\rangle^2)$ being less than 1 (see figure 3 in [40]).
- Spin squeezing test is based on assumption that Bloch vector is on Bloch sphere, a result established since $\langle\widehat{S}_x\rangle$ and $\langle\widehat{S}_y\rangle$ shown to lie on Bloch sphere.
- Claimed entanglement of ca 170 atoms.
- Comment—spin squeezing in $\widehat{S}_z$ demonstrated (see (107)), so entanglement is established. An entangled state was of course already created by first $\pi/2$ pulse, and then modified via the collisional effects.

8.4. Gross et al [39]

- Stated emphasis—continuous variable entangled *twin-atom states*.
- System—Rb^{87} in several hyperfine states.
- Independent BECs of Rb^{87} trapped in separate wells in a optical lattice.
- Number of atoms macroscopic, ca few 100 in each well.
- Spin dynamics in Zeeman hyperfine states (2, 0), (1, $\pm$1).
- Initially have BEC in (2, 0) hyperfine state—acts as pump mode.
- Spin conserving collisional coupling to (1, $\pm$1) hyperfine states—which act as the two mode system.
- One boson created in each of (1, $\pm$1) hyperfine states with two bosons lost from (2, 0) hyperfine state due spin conserving collisions.
- OPA type situation associated with spin changing collisions with (1, $\pm$1) hyperfine states acting as idler, signal modes.
- Mean and variance of population difference between (1, +1) and (1, −1) hyperfine states measured. Total population also measured.
- Entanglement test is that if the variance in population difference is small, but that in the total boson number is large then the state is entangled (see (57) and (58)).
- Measurements (see figure 1(c) in [39]) show noise in population difference is small, but that in the total boson number is large.
- Further entanglement test is that if there is two mode quadrature squeezing then the state is entangled.
- Comment—number squeezing and two mode quadrature squeezing demonstrated and entanglement confirmed.

9. Discussion and summary of key results

The two accompanying papers are concerned with mode entanglement for systems of identical massive bosons and the relationship to spin squeezing and other quantum correlation effects. These bosons may be atoms or molecules as in cold quantum gases. The previous paper I [1] dealt with the general features of quantum entanglement and its specific definition in the case of systems of identical bosons. In defining entanglement for systems of identical massive particles, it was concluded that the single particle states or modes are the most appropriate choice for sub-systems that are distinguishable, that the general quantum states must comply both with the symmetrization principle and the SSR that forbid quantum superpositions of states with differing total particle number (global SSR compliance), and that in the separable states quantum superpositions of sub-system states with differing sub-system particle number (local SSR compliance) also do not occur [2]. The present paper II has examined possible tests for two mode entanglement based on the treatment of entanglement set out in paper I.

The present paper first defines *spin squeezing* in *two mode* systems for the original spin operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$, which are defined in terms of the original mode annihilation and creation operators $\widehat{a}$, $\widehat{b}$ and $\widehat{a}^\dagger$, $\widehat{b}^\dagger$. Spin squeezing for the *principal spin operators* $\widehat{J}_x$, $\widehat{J}_y$, $\widehat{J}_z$ for which the *covariance matrix* is diagonal, rather than via the original spin operators is then discussed. It is seen that the two sets of spin operators are related via a rotation operator and the principal spin operators are given in terms of *new mode operators* $\widehat{c}$, $\widehat{d}$ and $\widehat{c}^\dagger$, $\widehat{d}^\dagger$, with $\widehat{c}$, $\widehat{d}$ obtained as linear combinations of the original mode operators $\widehat{a}$, $\widehat{b}$ and hence defining two new modes. Finally, we consider spin squeezing in the context of *multi-mode* systems.

The consequence for the case of two mode systems of identical bosons of the present approach to defining entangled states is that spin squeezing in *any* of the spin operators $\widehat{S}_x$, $\widehat{S}_y$ or $\widehat{S}_z$ *requires* entanglement of the original modes $\widehat{a}$, $\widehat{b}$. Similarly, spin squeezing in *any* of the new spin operators $\widehat{J}_x$, $\widehat{J}_y$ or $\widehat{J}_z$ requires entanglement of the new modes $\widehat{c}$, $\widehat{d}$. The full proof of these results has been presented here. A typical and *simple spin squeezing* test for entanglement is $\langle\Delta\widehat{S}_x^2\rangle < |\langle\widehat{S}_z\rangle|/2$ or $\langle\Delta\widehat{S}_y^2\rangle < |\langle\widehat{S}_z\rangle|/2$. We also found a simple *Bloch vector* test $\langle\widehat{S}_x\rangle \neq 0$ or $\langle\widehat{S}_y\rangle \neq 0$. It was noted that though spin squeezing requires entanglement, the opposite is not the case and the *NOON* state provided an example of an entangled physical state that is not spin squeezed. Also, the *binomial state* provided an example of a state that is entangled and spin squeezed for one choice of mode sub-systems but is non-entangled and not spin squeezed for another choice. The *relative phase state* provided an example that is entangled for new modes $\widehat{c}$, $\widehat{d}$ and is highly spin squeezed in $\widehat{J}_y$ and very un-squeezed in $\widehat{J}_x$. We then showed (see appendix D) that in certain *multi-mode* cases, spin squeezing in any spin component confirmed entanglement. In the multi-mode case this test applied in the bipartite case (*Case 1*) where the *two sub-systems* each consisted of all the modes $\widehat{a}_i$ *or* all the modes $\widehat{b}_i$ or in the *single modes* case (*Case 2*)where there were $2n$ sub-systems consisting of all the modes $\widehat{a}_i$ *and* all the modes $\widehat{b}_i$. For the *mode pairs* case (*Case 3*) where there were n sub-systems consisting of all the pairs of modes $\widehat{a}_i$ and $\widehat{b}_i$, a spin squeezing entanglement test was found in the situation where for separable states each mode pair involved a *single boson*. The connection between spin squeezing and entanglement was regarded as well-known, but up to now the only existing proofs were based on non-entangled states that disregarded either the symmetrization principle or the sub-system SSRs, placing the connection between spin squeezing and entanglement on a somewhat shaky basis. On the other hand, the proof given here is based on a definition of non-entangled (and hence entangled) states that is compatible with both these requirements.

There are several papers that have obtained *different tests* for whether a state is entangled from those involving squeezing for *spin operators*, the proofs often being based on a definition of non-entangled states that ignores the sub-system SSR. Hillery *et al* [22] obtained criteria of this type, such as the *spin variance* entanglement test $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle < \frac{1}{2}\langle\widehat{N}\rangle$. The proof of this test has also been set out here, and the test is also seen to be valid if the non-entangled state definition is consistent with the SSR. The test $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle < |\langle\widehat{S}_z\rangle|$ suggested by the requirement that $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle \geqslant |\langle\widehat{S}_z\rangle|$ for non-entangled states—since both $\langle\Delta\widehat{S}_x^2\rangle \geqslant |\langle\widehat{S}_z\rangle|/2$ and $\langle\Delta\widehat{S}_y^2\rangle \geqslant |\langle\widehat{S}_z\rangle|/2$ is of no use, since $\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle \geqslant |\langle\widehat{S}_z\rangle|$ for all states. However as previously noted, showing that either $\langle\Delta\widehat{S}_x^2\rangle < |\langle\widehat{S}_z\rangle|/2$ or $\langle\Delta\widehat{S}_y^2\rangle < |\langle\widehat{S}_z\rangle|/2$—or the analogous tests for other pairs of spin operators—already provides a test for the entanglement of the original modes $\widehat{a}$, $\widehat{b}$. This test is a different test for entanglement than that of Hillery *et al* [22]. In fact the case of the *relative phase eigenstate* is an example of an entangled state in which the simple spin squeezing test for entanglement *succeeds* whereas that of Hillery *et al* [22] *fails*. The consequences of applying both the simple spin squeezing and the Hillery spin operator test were examined (see Appendix C) with the aim of seeing whether the results could determine whether or not the local particle number SSR applied to separable states. The conclusion was negative as all outcomes were consistent with both possibilities. In addition, the Hillery spin variance test was also shown (see Appendix D) to apply to the multi-mode situation, in the Cases 1 and 2 described above, but did not apply in Case 3. Other entanglement tests of Benatti *et al* [29] involving variances of two mode spin operators were also found to apply for identical bosons.

The present paper also considered *correlation tests* for entanglement. Inequalities found by Hillery *et al* [32] for non-entangled states which also do not depend on whether non-entangled states satisfy the SSR include $|\langle(\widehat{a})^m(\widehat{b}^\dagger)^n\rangle|^2 \leqslant \langle(\widehat{a}^\dagger)^m(\widehat{a})^m(\widehat{b}^\dagger)^n(\widehat{b})^n\rangle$, giving a valid *strong correlation* test $|\langle(\widehat{a})^m(\widehat{b}^\dagger)^n\rangle|^2 > \langle(\widehat{a}^\dagger)^m(\widehat{a})^m(\widehat{b}^\dagger)^n(\widehat{b})^n\rangle$ for an entangled state. However, with entanglement defined as in the present paper we have $|\langle (\hat{a})^m (\hat{b}^\dagger)^n \rangle|^2 = 0$ for a non-entangled state, so we have also proved a *weak correlation* test for entanglement in the form $|\langle (\hat{a})^m (\hat{b}^\dagger)^n \rangle|^2 > 0$. This test is less stringent than the strong correlation test of Hillery *et al* [32], as $|\langle (\hat{a})^m (\hat{b}^\dagger)^n \rangle|^2$ is then required to be larger. In all these cases. For $n \neq m$ none of these cases are of interest since for global SSR compliant states $\langle (\hat{a})^m (\hat{b}^\dagger)^n \rangle$ would be zero. In the cases where $n = m$ we show that *all* the correlation tests can be expressed in terms of *spin operators*, so they reduce to tests involving powers of spin operators. For the case $n = m = 1$ the weak correlation test is the same as the Bloch vector test.

Work by other authors on bipartite entanglement tests has also been examined here. He *et al* [23, 25] considered a *four mode* system, with two modes localized in each well of a double well potential. If the two sub-systems A and B each consist of two modes—with $\hat{a}_1$, $\hat{a}_2$ as sub-system A and $\hat{b}_1$, $\hat{b}_2$ as sub-system B, then tests of bipartite entanglement of the two sub-systems of the Hillery [32] type $|\langle (\hat{a}_i)^m (\hat{b}_j^\dagger)^n \rangle|^2 > \langle (\hat{a}_i^\dagger)^m (\hat{a}_i)^m (\hat{b}_j^\dagger)^n (\hat{b}_j)^n \rangle$ for any $i, j = 1, 2$ or involving local spin operators $|\langle \hat{S}_+^A \hat{S}_-^B \rangle|^2 > \langle \hat{S}_+^A \hat{S}_-^A \hat{S}_+^B \hat{S}_-^B \rangle$ apply. Raymer *et al* [26] have also considered such a four mode system and derived bipartite entanglement tests such as $\langle \Delta(\hat{S}_x^A \pm \hat{S}_x^B)^2 \rangle + \langle \Delta(\hat{S}_y^A \mp \hat{S}_y^B)^2 \rangle < |\langle \hat{S}_z \rangle|$ that involve local spin operators for the two sub-systems.

We also considered the work of Sørensen *et al* [14], who showed that spin squeezing is a test for a state being entangled, but defined non-entangled states for identical particle systems (such as BECs) in a form that is *inconsistent* with the symmetrization principle—the sub-systems being regarded as individual identical particles. However, the treatment of Sørensen *et al* [14] can be modified to apply to a system of identical bosons if the particle index i is *re-interpreted* as specifying different modes, for example modes localized on optical lattice sites $i = 1, 2, .., n$ or localized in momentum space. With two single particle states $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ with annihilation operators a_i, b_i available on each site, there would then be $2n$ modes involved, but spin operators can still be defined. This is just a particular case of the multi-mode situation described above. If the definitions of non–entangled and entangled states in the present paper are applied, it can be shown that spin squeezing in either of the spin operators $\hat{S}_x$ or $\hat{S}_y$ requires entanglement of *all* the original modes $\hat{a}_i$, $\hat{b}_i$ (Case 1, above). Alternatively, if the sub-systems are *pairs* of modes $\hat{a}_i$, $\hat{b}_i$ *and* the sub-system density operators $\hat{\rho}_R^i$ were restricted to states with exactly *one boson*, then it can be shown that spin squeezing in $\hat{S}_z$ requires entanglement of all the pairs of modes (Case 3, above). With this restriction the pair of modes $\hat{a}_i$, $\hat{b}_i$ behave like *distinguishable* two state particles, which was essentially the case that Sørensen *et al* [14] implicitly considered. This type of entanglement is a multi-mode entanglement of a special type—since the modes $\hat{a}_i$, $\hat{b}_i$ may themselves be entangled there is an 'entanglement of entanglement'. So with either of these revisions, the work of Sørensen *et al* [14] could be said to show that spin squeezing requires entanglement. However, neither of these revisions really deals with the case of entanglement in *two mode* systems, and here the proof given in this paper showing that spin squeezing in $\hat{S}_z$ requires entanglement of the two modes provides the justification of this result *without* treating identical particles as distinguishable sub-systems. Sørensen and Mølmer [30] have also deduced an inequality involving $\langle \Delta \hat{S}_x^2 \rangle$ and $|\langle \hat{S}_z \rangle|$ for states where $\langle \hat{S}_x \rangle = \langle \hat{S}_y \rangle = 0$ based on just the Heisenberg uncertainty principle. This is useful in terms of confirming that states do exist that are spin squeezed still conform to this principle.

Entanglement tests involving quadrature variables have also been published, so we have also examined these. Duan *et al* [33], Toth *et al* [34] devised a test for entanglement based on the sum of the *quadrature variances* $\langle \Delta(\hat{x}_A \pm \hat{x}_B)^2 \rangle + \langle \Delta(\hat{p}_A \mp \hat{p}_B)^2 \rangle \geqslant 2$ for separable states, which involve quadrature components $\hat{x}_A$, $\hat{p}_A$, $\hat{x}_B$, $\hat{p}_B$ constructed from the original mode annihilation, creation operators for modes A, B. Their conclusion that if the quadrature variances sum is less than 2 then the state is entangled is valid both for the present definition of entanglement and for that in which the application of the SSRe is ignored. However, for quantum states for systems of identical bosons that are global SSR compliant $\langle \Delta(\hat{x}_A \pm \hat{x}_B)^2 \rangle + \langle \Delta(\hat{p}_A \mp \hat{p}_B)^2 \rangle = 2 + 2\langle \hat{N} \rangle$ for *all* such states—both separable and entangled, and although this is consistent with [33, 34] we have concluded that the quadrature variance test can *never* confirm entanglement. A more general test [35] involving quadrature operators $\hat{X}_A^\theta$, $\hat{X}_B^\theta$ required showing that $\langle \hat{X}_A^\theta \hat{X}_B^\phi \rangle \neq 0$. This was shown to be equivalent to showing that $\langle \hat{S}_x \rangle \neq 0$ or $\langle \hat{S}_y \rangle \neq 0$, the *Bloch vector* or *weak correlation* test. A *two mode quadrature squeezing* test was also obtained, but found to be less useful than the *Bloch vector* test.

Overall then, all of the *entanglement tests* (spin squeezing and other) in the other papers discussed here are *still valid* when reconsidered in accord with the definition of entanglement based on the symmetrization and SSRs, though in one case Sørensen *et al* [14] a re-definition of the sub-systems is required to satisfy the symmetrization principle. However, *further* tests for entanglement are obtained in the present paper based on non-entangled states that are consistent with the symmetrization and SSRs. In some cases they are less stringent—the correlation test in equation (121) being easier to satisfy than that of Hillery *et al* [32] in equation (124). The tests introduced here are certainly *different* to others previously discovered.

The theory for a simple *two mode interferometer* was then presented and it was shown that such an interferometer can be used to measure the mean values and covariance matrix for the spin operators involved in entanglement tests for the two mode bosonic system. The treatment was also generalized (see Appendix M) to *multi-mode* interferometry. The interferometer involved a *pulsed classical field* characterized by a *phase* variable ϕ and an *area* variable $2s = \theta$ defined by the time integral of the field amplitude, and leads to a coupling between the two modes. For simplicity the center frequency was chosen to be *resonant* with the mode transition frequency. Measuring the mean and variance of the *population difference* between the two modes for the *output* state of the interferometer for various choices of ϕ and θ enabled the mean values and covariance matrix for the spin operators for the *input* quantum state of the two mode system to be determined. More complex interferometers were seen to involve combinations of simple interferometers separated by time intervals during which *free evolution* of the two mode system can occur, including the effect of *collisions*.

Experiments have been carried out demonstrating that spin squeezing occurs, which according to theory requires entanglement. An analysis of these experiments has been presented here. However, since no results for entanglement *measures* are presented or other *independent* tests for entanglement carried out, the entanglement presumably created in the experiments has not been independently confirmed.

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Appendix

The following appendices can be found in the supplementary data here, stacks.iop.org/ps/92/023005/mmedia

References to the cited articles [43–48] can be found within the supplementary data here stacks.iop.org/ps/92/023005/mmedia.

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