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FROM NONLINEAR TO LINEARIZED ELASTICITY VIA Γ-CONVERGENCE: THE CASE OF MULTIWELL ENERGIES SATISFYING WEAK COERCIVITY CONDITIONS

VIRGINIA AGOSTINIANI, TIMOTHY BLASS, AND KONSTANTINOS KOUMATOS

Abstract. Linearized elasticity models are derived, via Γ-convergence, from suitably rescaled nonlinear energies when the corresponding energy densities have a multiwell structure and satisfy a weak coercivity condition, in the sense that the typical quadratic bound from below is replaced by a weaker p bound, $1 < p < 2$, away from the wells. This study is motivated by, and our results are applied to, energies arising in the modeling of nematic elastomers.

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Linearized elasticity, Γ-convergence, nematic elastomers

1. Introduction

Consider a homogeneous and hyperelastic body occupying in its reference configuration a bounded domain $\Omega \subset \mathbb{R}^d$. Deformations of the body are described by mappings $v : \Omega \to \mathbb{R}^d$, where $v(x)$ denotes the deformed position of the material point $x \in \Omega$. The total elastic energy corresponding to the deformation v is given by

$$\int_\Omega W(\nabla v(x))dx,$$

where $\nabla v \in \mathbb{R}^{d\times d}$ is the deformation gradient and $W : \mathbb{R}^{d\times d} \to \mathbb{R}$ is a frame-indifferent energy density associated with the material. More generally, we consider energies of the form

$$\int_\Omega W(\nabla v(x))\,dx - \int_\Omega l(x)\cdot u(x)\,dx,$$

where $u(x) = v(x) - x$ is the displacement and $l(x)$ is an external (dead) load at $x \in \Omega$, so that the term $\int_\Omega l \cdot u\,dx$ accounts for the work performed by the applied loads.

Let us illustrate the idea behind the passage from nonlinear to linearized elasticity. Suppose that W is C^2 near the identity, nonnegative (up to additive constants), and vanishing precisely on $SO(d)$. In the absence of external loads, the deformation $v(x) = x$ is an equilibrium state and it is natural to expect that small external loads εl result in small displacements εu, where $\varepsilon > 0$ is a small parameter. The associated energy then becomes

$$\int_\Omega W(I + \varepsilon\nabla u(x))\,dx - \varepsilon^2 \int_\Omega l(x)\cdot u(x)\,dx, \tag{1.1}$$

where $I \in \mathbb{R}^{d\times d}$ is the identity matrix.

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Assuming that $u \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ and rescaling (1.1) by $1/\varepsilon^2$, the limit as $\varepsilon \to 0$ yields

$$\frac{1}{2}\int_\Omega D^2W(I)[e(u)]^2\,dx - \int_\Omega l(x)\cdot u(x)\,dx, \tag{1.2}$$

where $e(u) := [\nabla u + (\nabla u)^T]/2$. Recall that the quadratic form $M \mapsto D^2W(I)[M]^2$ acts only on the symmetric part of M, due to frame-indifference.

Note that this argument is restricted to $u \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ and does not entail whether minimizers u_ε of the rescaled nonlinear energies

$$\frac{1}{\varepsilon^2}\int_\Omega W(I + \varepsilon\nabla u(x))\,dx - \int_\Omega l(x)\cdot u(x)\,dx,$$

subject to suitable boundary data, actually converge to the minimizer of the limiting linearized[1] problem (1.2), under the same boundary data. The rigorous derivation of the linearized elastic formula (1.2) from nonlinear elasticity was provided in [12] via Γ-convergence, under the condition

$$W(F) \geq c\,\mathrm{dist}^2(F, SO(d)). \tag{1.3}$$

In this paper, we derive linearized models from nonlinear energies with a multi-well structure, i.e. W is minimized on a set $\mathcal{U}$ of the form $SO(d)U$, U ranging in a compact subset of positive definite, symmetric matrices. Also, we weaken condition (1.3) (with $SO(d)$ replaced by $\mathcal{U}$) to

$$W(F) \geq c\,\mathrm{dist}^p(F, \mathcal{U}),$$

$1 < p < 2$, for F away from $\mathcal{U}$; the coercivity remaining quadratic near $\mathcal{U}$.

Energies of this type arise naturally in a large class of compressible models for rubber-like materials, including nematic elastomers, the latter being materials consisting of networks of polymer chains with embedded liquid crystalline molecules. In [14], some nonlinear compressible models for nematic elastomers are considered together with their formally derived small-strain theories. These nonlinear models satisfy our assumptions and our results rigorously justify their geometrically linear counterparts (see Theorem 3.1).

In order to derive small-strain limiting theories, we introduce a small parameter ε and we consider a family of densities $\{W_\varepsilon\}$ with corresponding energy wells

$$\mathcal{U}_\varepsilon := SO(d)\{U_\varepsilon = U_\varepsilon^T = I + \varepsilon U + o(\varepsilon) : U \in \mathcal{M}\}, \tag{1.4}$$

$\mathcal{M}$ being a compact subset of symmetric matrices and $o(\varepsilon)$ being uniform with respect to $U \in \mathcal{M}$. We assume that

$$W_\varepsilon \geq c\,\mathrm{dist}^2(\cdot, \mathcal{U}_\varepsilon) \text{ near } \mathcal{U}_\varepsilon, \qquad W_\varepsilon \geq C\,\mathrm{dist}^p(\cdot, \mathcal{U}_\varepsilon) \text{ away from } \mathcal{U}_\varepsilon, \tag{1.5}$$

and we investigate the limiting behavior, as $\varepsilon \to 0$, of the rescaled functionals

$$\mathscr{E}_\varepsilon(u) := \frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(I + \varepsilon\nabla u(x))\,dx - \int_\Omega l(x)\cdot u(x)\,dx$$

and their (almost) minimizers. For a discussion on the choice of the various scalings, the reader is referred to [23].

In view of the coercivity assumption, the natural ambient space is $W^{1,p}(\Omega, \mathbb{R}^d)$, where one can prove equicoercivity of the functionals $\mathscr{E}_\varepsilon$. This compactness, coupled with a Γ-convergence result, allows us to prove that, under suitable boundary data, the infima of $\mathscr{E}_\varepsilon$ over $W^{1,p}(\Omega, \mathbb{R}^d)$ converge to the infimum of

$$\mathscr{E}(u) := \int_\Omega V(e(u(x)))dx - \int_\Omega l(x)\cdot u(x)dx$$

[1]Note that the limiting energy density in (1.2) is quadratic and corresponds to a linear stress–strain relation. For multiwell energies this is not the case and one may only speak of *geometrically linear* models. Thus, the term linearized is preferred over the term linear.

over $W^{1,2}(\Omega,\mathbb{R}^d)$, under the same boundary conditions. The linearized energy density V is obtained as the limit $\varepsilon\to 0$ of the quantities $\varepsilon^{-2}W(I+\varepsilon\cdot)$, whenever this limit is uniform on compact subsets of symmetric matrices. Moreover, sequences of almost minimizers of the functionals $\mathscr{E}_\varepsilon$ converge to a minimizer of the relaxation of $\mathscr{E}$ in $W^{1,p}(\Omega,\mathbb{R}^d)$. This is the content of Theorem 2.1.

We remark that the first attempt to rigorously justify the passage from nonlinear to linearized elasticity in the case of multiwell energy densities was due to B. Schmidt in [23], where the author assumes the standard quadratic coercivity condition (corresponding to (1.5) with $p=2$) [2]. In the same paper Schmidt applies his results to discuss the validity of the so-called KRS model [18] for crystalline solids which can be thought of as a formal linearization of nonlinear theories for solid-to-solid phase transitions (see [5] and [19]). The theory developed in [23] was later applied in [2] to justify certain linearized models for nematic elastomers with quadratic growth. To include other natural compressible models for nematic elastomers, we extend the results of [23] to the case $1<p<2$. Some of the proofs rely on techniques introduced in [4] where the case of single well energies satisfying the weak coercivity condition is treated.

The paper is organized as follows: in Section 2, we introduce all the ingredients and state our main results. The models for nematic elastomers under consideration are described in detail in Section 3 where the results of Section 2 are applied. Section 4 is devoted to the proofs of the main statements. In Section 5 we prove that it is possible to provide a Young measure representation for the limiting functional, as well as prove the strong convergence of sequences of almost minimizers under strong convexity assumptions on the limiting density V. The paper concludes with an Appendix where various already established results are gathered, along with their proofs, for the convenience of the reader.

2. Main results

The sets of matrices we work with are $\mathbb{R}^{d\times d}$ ($d\times d$ real matrices), $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ (symmetric matrices), $SO(d)$ (rotations). Here and throughout, $c>0$ denotes a generic constant which might differ in each instance. We denote by id the identity function on $\mathbb{R}^d$ and by $I\in\mathbb{R}^{d\times d}$ the identity matrix. For every $F\in\mathbb{R}^{d\times d}$, $\operatorname{sym}F:=\frac{F+F^T}{2}$.

Let $W_\varepsilon:\mathbb{R}^{d\times d}\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{\infty\}$ be a family of frame-indifferent multiwell energy densities with a corresponding set of wells $\mathcal{U}_\varepsilon$ given by (1.4). Note that the matrices $U_\varepsilon\in\mathcal{U}_\varepsilon$ are positive definite for every ε small enough. We also assume that the energies W_ε are measurable, continuous in an ε-independent neighborhood of the identity and satisfy the following coercivity condition:

$$W_\varepsilon(F)\geq c g_p(\operatorname{dist}(F,\mathcal{U}_\varepsilon)), \tag{2.1}$$

for all $F\in\mathbb{R}^{d\times d}$ and for a constant $c>0$ independent of ε, where for some $1<p\leq 2$, $g_p:[0,\infty)\to\mathbb{R}$ is given by:

$$g_p(t)=\begin{cases}\frac{t^2}{2}, & t\in[0,1]\\[2ex] \frac{t^p}{p}+\left(\frac{1}{2}-\frac{1}{p}\right), & t\in[1,\infty).\end{cases} \tag{2.2}$$

[2]Note that in [23] the author also assumes that the set $\mathcal{M}$ appearing in (1.4) consists of a finite number of symmetric matrices. However, the same results extend to the case of a compact set $\mathcal{M}$ without changing the proofs.

To retain physicality, though not required for the proofs, we impose the additional condition that

$$W_\varepsilon(F) \to +\infty, \quad \text{as} \quad \det F \to 0,$$
$$W_\varepsilon(F) = +\infty, \quad \text{if} \quad \det F \leq 0.$$

The reference configuration is represented by a bounded and Lipschitz domain $\Omega \subset \mathbb{R}^d$ and, to incorporate the boundary data, we fix $h \in W^{1,\infty}(\Omega, \mathbb{R}^d)$, a subset $\partial_D\Omega \subseteq \partial\Omega$ of positive surface measure, and introduce for every $1 < p \leq 2$ the set

$$W_h^{1,p} := \{u \in W^{1,p}(\Omega, \mathbb{R}^d) : Tu = Th \text{ on } \partial_D\Omega\}, \tag{2.3}$$

where T stands for the trace operator. We require $\partial_D\Omega$ to have *Lipschitz boundary* in $\partial\Omega$ according to [4, Definition 2.1]. This condition implies that $W_h^{1,p}$ agrees with the closure of $W_h^{1,\infty}$ in $W^{1,p}(\Omega, \mathbb{R}^d)$ (see [4, Proposition A.2]). We use this equivalence in the proof of Theorem 2.3 below.

A continuous and linear functional $\mathscr{L} : W^{1,p}(\Omega, \mathbb{R}^d) \to \mathbb{R}$, with p as in (2.1), represents the applied loads. It is in principle a function of the deformation v, but it enters the expression of the total energy of the system only as $\mathcal{L}(u)$, where $u(x) = v(x) - x$ is the displacement associated with the deformation v. This is because the total energy can be renormalized by $-\mathscr{L}(id)$, in view of the linearity of $\mathscr{L}$.

The problem under investigation is to understand the behavior, as $\varepsilon \to 0$, of the infimum of the total energy appropriately rescaled by $1/\varepsilon^2$:

$$\frac{1}{\varepsilon^2} \int_\Omega W_\varepsilon(I + \varepsilon\nabla u)dx - \mathscr{L}(u)$$

subject to the boundary data h. The analysis is thus based on the rescaled quantities $W_\varepsilon(I + \varepsilon F)/\varepsilon^2$, whose limit as $\varepsilon \to 0$ depends only on the symmetric part of F, due to frame indifference. Thus, we consider the rescaled densities

$$V_\varepsilon(E) := \frac{1}{\varepsilon^2} W_\varepsilon(I + \varepsilon E), \tag{2.4}$$

defined for every $E \in \mathbb{R}^{d\times d}_{\rm sym}$, or, equivalently, their extensions $f_\varepsilon : \mathbb{R}^{d\times d} \to \mathbb{R}$ given by

$$f_\varepsilon(F) := V_\varepsilon(\operatorname{sym} F). \tag{2.5}$$

Assume that $V_\varepsilon \to V$ uniformly on compact subsets of $\mathbb{R}^{d\times d}_{\rm sym}$ for some $V : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}$. Note that this is equivalent to asking that $f_\varepsilon \to f$ uniformly on compact subsets of $\mathbb{R}^{d\times d}$ where $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ is the extension of V given by

$$f(F) := V(\operatorname{sym} F). \tag{2.6}$$

Also, we remark that V satisfies the growth condition

$$V(E) \leq c(1 + |E|^2)$$

if and only if $f(F) \leq c(1 + |F|^2)$.

Observe that, in view of the growth condition (2.1) and Lemma 4.4, if $V(E) = 0$ then $E \in \mathcal{M}$, where $\mathcal{M}$ appears in definition (1.4). The following theorem is our main result.

Theorem 2.1. *Let $1 < p \leq 2$, suppose that $f_\varepsilon \to f$ uniformly on compact subsets of $\mathbb{R}^{d\times d}$, and that f satisfies $0 \leq f(F) \leq c(1 + |F|^2)$ for every $F \in \mathbb{R}^{d\times d}$ and some constant $c > 0$. If*

$$m_\varepsilon := \inf_{u \in W_h^{1,p}} \left\{ \frac{1}{\varepsilon^2} \int_\Omega W_\varepsilon(I + \varepsilon\nabla u)dx - \mathscr{L}(u) \right\},$$

and if $\{u_\varepsilon\}$ *is a sequence such that*

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(I+\varepsilon\nabla u_\varepsilon)dx - \mathscr{L}(u_\varepsilon)\right\} = \lim_{\varepsilon\to 0} m_\varepsilon, \tag{2.7}$$

then, up to a subsequence, $u_\varepsilon \rightharpoonup u$ *in* $W^{1,p}(\Omega,\mathbb{R}^d)$*, where* u *is a solution to the minimum problem*

$$m := \min_{u\in W^{1,2}_h}\left\{\int_\Omega f^{qc}(\nabla u)dx - \mathscr{L}(u)\right\}. \tag{2.8}$$

Moreover, $m_\varepsilon \to m$*.*

The integrand f^{qc} obtained in the limit is the *quasiconvexification* of f. The corresponding notion for V is the *quasiconvexification on linear strains* (see Subsection 6.2 for definitions), and we denote it by V^{qce}. Although we have the equality $f^{qc}(F) = V^{qce}(\operatorname{sym} F)$ for every $F \in \mathbb{R}^{d\times d}$ (see Proposition 6.4), we prefer to retain both the notation f^{qc} and V^{qce}, because while our proofs seem more natural in terms of f^{qc}, some results are more easily stated in terms of V^{qce}.

Remark 2.1. Note that condition (2.1) and the definition of V as the uniform limit on compact subsets of $\mathbb{R}^{d\times d}_{\text{sym}}$ of V_ε in (2.4), yields the existence of constants $C_1, C_2 > 0$ such that

$$V(E) \geq C_1|E|^2 - C_2.$$

Indeed, for ε sufficiently small, $\operatorname{dist}(I+\varepsilon E, \mathcal{U}_\varepsilon) \leq 1$ so that by the definition of g_p

$$\begin{aligned} V(E) &\geq \limsup_{\varepsilon\to 0}\frac{c}{2\varepsilon^2}\operatorname{dist}^2(I+\varepsilon E,\mathcal{U}_\varepsilon) \\ &\geq C_1\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\operatorname{dist}^2(I+\varepsilon E, SO(d)) - C_2 = C_1|E|^2 - C_2, \end{aligned}$$

where in the last equality we have used (4.31).

The proof of Theorem 2.1 is based on two intermediate results: a compactness result following from equicoercivity, and a Γ-convergence result. In order to state them, we define the approximate functionals $\mathscr{E}_\varepsilon : W^{1,p}(\Omega,\mathbb{R}^d) \to \overline{\mathbb{R}}$ by

$$\mathscr{E}_\varepsilon(u) := \begin{cases} \frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(I+\varepsilon\nabla u)\,dx, & u \in W^{1,p}_h \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.9}$$

and the limiting functional $\overline{\mathscr{E}} : W^{1,p}(\Omega,\mathbb{R}^d) \to \overline{\mathbb{R}}$ by

$$\overline{\mathscr{E}}(u) := \begin{cases} \int_\Omega f^{qc}(\nabla u)\,dx, & u \in W^{1,2}_h \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.10}$$

Proposition 2.2 (Equicoercivity)**.** *Let* $1 \leq p \leq 2$*. There exists a constant* $C = C(\Omega, p, \partial_D\Omega, h) > 0$ *such that*

$$\int_\Omega |\nabla u|^p dx \leq C(1 + \mathscr{E}_\varepsilon(u))$$

for every $u \in W^{1,p}_h$ *and every* ε *sufficiently small.*

This result allows us to deduce that, in the case $1 < p \leq 2$, if we have a sequence $\{u_\varepsilon\}$ of almost minimizers, that is $\{u_\varepsilon\}$ satisfies (2.7), then, up to a subsequence, $u_\varepsilon \rightharpoonup u \in W^{1,p}_h$. By standard Γ-convergence arguments, Theorem 2.3 below then implies that u is indeed a solution of the minimum problem (2.8).

Theorem 2.3 (Γ-convergence)**.** *Let* $1 < p \leq 2$*. Under the hypotheses of Theorem 2.1, the sequence of functionals* $\{\mathscr{E}_\varepsilon\}$ *Γ-converges to* $\overline{\mathscr{E}}$ *with respect to the weak topology of* $W^{1,p}(\Omega,\mathbb{R}^d)$*.*

To obtain this result, the requirement $h \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ is sharp in the sense that there are some particular $h \in W^{1,q}(\Omega, \mathbb{R}^d)$, with $2 \le q < \infty$ such that the Γ-convergence does not hold unless the energy densities satisfy suitable bounds from above which are not natural in this context (see [4, Remark 2.7]).

Remark 2.2 (Relaxation). Notice that from Theorem 2.3 we have also obtained, as a by-product, that $\overline{\mathscr{E}}$ is the sequentially weak lower semicontinuous envelope in $W^{1,p}(\Omega, \mathbb{R}^d)$ of the functional $\mathscr{E} : W^{1,p}(\Omega, \mathbb{R}^d) \to \overline{\mathbb{R}}$ defined by

$$\mathscr{E}(u) := \begin{cases} \int_\Omega f(\nabla u)\, dx, & u \in W^{1,2}_h \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

Indeed, by standard Γ-convergence results, the functional $\overline{\mathscr{E}}$ is weakly lower semicontinuous in $W^{1,p}(\Omega, \mathbb{R}^d)$. Hence, whenever $u_j \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^d)$, we have that

$$\overline{\mathscr{E}}(u) \le \liminf_{j\to\infty} \overline{\mathscr{E}}(u_j) \le \liminf_{j\to\infty} \mathscr{E}(u_j),$$

since $\overline{\mathscr{E}} \le \mathscr{E}$. On the other hand, to prove the existence of a relaxing sequence it is enough to consider $u \in W^{1,2}_h$ and then standard relaxation results (see Theorem 6.2) for the functional $\mathscr{E}$ restricted to $W^{1,2}(\Omega, \mathbb{R}^d)$ provide the required sequence. In particular, we have that $\min_{W^{1,2}_h} \overline{\mathscr{E}} = \inf_{W^{1,2}_h} \mathscr{E}$. Note also that $\overline{\mathscr{E} - \mathscr{L}} = \overline{\mathscr{E}} - \mathscr{L}$.

Next we present Corollary 2.4, analogous to [23, Corollary 2.8]. We begin by introducing some notation. Let $\mathcal{Q}$ denote the set of quasiconvex functions from $\mathbb{R}^{d\times d}$ to $\mathbb{R}$ and for $1 \le q < \infty$ let $\mathcal{Q}_q$ denote the set of functions $f \in \mathcal{Q}$ such that $0 \le f(F) \le C(1+|F|^q)$ for every F and some $C > 0$. For a compact set $K \in \mathbb{R}^{d\times d}$, the *strong q-quasiconvex hull* of K is

$$\mathbf{Q}_q K := \left\{ F \in \mathbb{R}^{d\times d} \, : \, f(F) \le \sup_{G\in K} f(G) \text{ for every } f \in \mathcal{Q}_q \right\}. \tag{2.11}$$

Recall that the *quasiconvex hull* QK of K is defined as the right-hand side of (2.11) with $\mathcal{Q}$ in place of $\mathcal{Q}_q$. On the other hand, the *weak q-quasiconvex hull* $Q_q K$ of K is the zero-level set of the quasiconvexification of the function $F \mapsto \mathrm{dist}^q(F, K)$. Finally, we define the sets $Q^e K$, $\mathbf{Q}^e_q K$, and $Q^e_q K$ analogously in terms of quasiconvexity on linear strains.

Corollary 2.4. *Suppose $\partial_D \Omega = \partial\Omega$, and $h(x) = Fx$ for a fixed $F \in \mathbb{R}^{d\times d}$. If $\{u_\varepsilon\} \subset W^{1,p}_h$ satisfies*

$$\liminf_{\varepsilon\to 0} \left\{ \frac{1}{\varepsilon^2} \int_\Omega W_\varepsilon(I + \varepsilon\nabla u_\varepsilon) dx \right\} = 0,$$

then $\mathrm{sym}\, F \in \{V^{qce} = 0\} \subseteq Q^e_2\mathcal{M}$.

The idea is that at low energy scales, i.e. $\varepsilon^{-2} \int_\Omega W_\varepsilon \ll 1$, restrictions are imposed on the possible boundary data F so that they are compatible with the wells $U \in \mathcal{M}$. For example, it is straightforward to show (see the proof of Corollary 2.4) that the data F must satisfy $\mathrm{sym}\, F \in \{V^{qce} = 0\}$. However, such restrictions can be improved and are typically expressed in terms of some quasiconvex hull of the wells. In this case, the appropriate restriction appears to be $\mathrm{sym}\, F \in Q^e_2\mathcal{M}$. Corollary 2.4 asserts that $\{V^{qce} = 0\} \subseteq Q^e_2\mathcal{M}$, so that the restriction is mild; it would be interesting to know if further restrictions could be imposed on F.

We remark that for a general V it is not known whether $\{V^{qce} = 0\} = Q^e\{V = 0\}$, but it is always true (and it is easy to check) that $Q^e\{V = 0\} \subseteq \{V^{qce} = 0\}$. On the other hand, in [25, Theorem 4] it is proved that

$$\mathbf{Q}^e_q \tilde{K} = Q^e_q \tilde{K} = Q^e_1 \tilde{K}, \quad \text{for every } q \in [1, \infty),$$

for every compact $\tilde{K} \in \mathbb{R}^{d\times d}_{\rm sym}$ but it is not known whether $Q^e_q\tilde{K} = Q^e\tilde{K}$ is true in general. The nonlinear version of these results is given by [24, Proposition 2.5] and we have that

$$QK = \mathbf{Q}_q K = Q_q K, \quad \text{for every } q \in [1,\infty),$$

for every compact $K \in \mathbb{R}^{d\times d}$.

Remark 2.3. For energies describing nematic elastomers (materials to which our results apply, see Section 3), more can be said for the geometrically linear as well as for the nonlinear case. Indeed, in the geometrically linear case, the fact that $\{V^{qce} = 0\} = Q^e\{V = 0\}$ is proved in [7] (see also [8]), while the nonlinear case $\{W^{qc} = 0\} = Q\{W = 0\}$ is due to [13].

Remark 2.4 (Inhomogeneous materials). We conclude this section by noting that all the results stated hold in the more general case of inhomogeneous materials, that is when the energy densities W_ε are also functions of $x \in \Omega$. In this case, our hypotheses can be reformulated in the following way: $W_\varepsilon : \Omega \times \mathbb{R}^{d\times d} \to [0,\infty]$ is measurable, $W_\varepsilon(x,\cdot)$ is continuous in an (ε, x)-independent neighborhood of the identity and frame-indifferent, and $W_\varepsilon(x,F) = 0$ if and only if $F \in \mathcal{U}_\varepsilon$ for a.e. $x \in \Omega$. Moreover,

$$W_\varepsilon(x,F) \geq c g_p(d(F,\mathcal{U}_\varepsilon))$$

for all $F \in \mathbb{R}^{d\times d}$, for a.e. $x \in \Omega$, and some $c > 0$ independent of ε and x. The functions $f_\varepsilon(F)$ are replaced by $f_\varepsilon(x,F)$ and we require that $f_\varepsilon \to f$ uniformly on $\Omega \times K$ for every compact $K \in \mathbb{R}^{d\times d}$, and that $f(x,F) \leq c(1 + |F|^2)$ for every $F \in \mathbb{R}^{d\times d}$ and some constant $c > 0$ independent of x. Finally, $f^{qc}(F)$ has to be replaced by $f^{qc}(x,F)$, where $f^{qc}(x,\cdot) = (f(x,\cdot))^{qc}$ for a.e. $x \in \Omega$.

3. Application to Nematic Elastomers

In this section, we consider the case $d = 3$, so all deformations are maps from $\mathbb{R}^3$ to $\mathbb{R}^3$. We use the notation $\mathrm{tr}F^2$ to denote the trace of the square of a matrix $F \in \mathbb{R}^{3\times 3}$, while $\mathrm{tr}^2 F$ stands for $(\mathrm{tr}F)^2$. The unit sphere of $\mathbb{R}^3$ is denoted by $\mathbb{S}^2$.

We begin by recalling that the standard *neo-Hookean* energy for incompressible deformations

$$W(F) = C\left(|F|^2 - 3\right), \quad \text{if } \det F = 1, \tag{3.1}$$

with $C > 0$, has a natural generalization to compressible strains (see [17])

$$W_{\rm comp}(F) = W((\det F)^{-1/3}F) + W_{\rm vol}(\det F) \tag{3.2}$$

$$= C\left(\frac{|F|^2}{(\det F)^{2/3}} - 3\right) + W_{\rm vol}(\det F), \quad \det F > 0. \tag{3.3}$$

The $1/3$ power in (3.2) is natural because $\det[(\det F)^{-1/3}F] = 1$, whenever $\det F > 0$. We assume the function $W_{\rm vol}$ satisfies the following natural properties:

$$\left.\begin{array}{l} W_{\rm vol} \in C^2((0,\infty),\mathbb{R}), \\ W_{\rm vol}(t) = 0 \text{ if and only if } t = 1, \\ W_{\rm vol}(t) \to +\infty, \text{ as } t \to 0^+, \\ W_{\rm vol}(t) \geq k\,t^2, \text{ for every } t \geq M > 0, \text{ for some } M, k > 0, \\ W''_{\rm vol}(1) > 0. \end{array}\right\} \tag{3.4}$$

In the condition $W''_{\rm vol}(1) > 0$, the strict inequality is important for our analysis to apply, as will be apparent later. An example of $W_{\rm vol}$ is $W_{\rm vol}(t) = t^2 - 1 - 2\log t$.

We note that if $W_{\rm comp}$ is given by (3.3) and $W_{\rm vol}$ satisfies (3.4), then $W_{\rm comp}(F) \geq 0$ and $W_{\rm comp}(F) = 0$ if and only if $F \in SO(3)$. This can be seen by using a standard inequality between arithmetic and geometric mean.

The transition from incompressible to compressible energies is the same for models of nematic elastomers. We begin by considering the standard energy density for modeling incompressible nematic elastomers given by

$$W(F) := \min_{n\in\mathbb{S}^2} W_n(F), \quad \text{if } \det F = 1,$$

where

$$W_n(F) := \frac{\mu}{2}\left[\operatorname{tr}(F^T L_n^{-1} F) - 3\right],$$
$$L_n := a^{2/3} n\otimes n + a^{-1/3}(I - n\otimes n),$$

and $\mu > 0$, $a > 1$ are constants. This density has been studied, e.g., in [2, 3, 13, 14]. Note that n is an eigenvector of L_n, with eigenvalue $a^{2/3}$. Any nonzero vector perpendicular to n is also an eigenvector of L_n, with eigenvalue $a^{-1/3}$. Hence, $\det L_n = 1$. It is straightforward to check that

$$L_n^\alpha = a^{2\alpha/3} n\otimes n + a^{-\alpha/3}(I - n\otimes n), \quad \text{for every } \alpha\in\mathbb{R}.$$

Note that W_n can be written in the neo-Hookean form

$$W_n(F) = \frac{\mu}{2}\left[\operatorname{tr}(F_n^T F_n) - 3\right], \quad F_n = L_n^{-\frac{1}{2}} F,$$

which shows that only the quantity F_n related to the deformation gradient F is responsible for the storage of energy. Generalizing this form of W_n, just as W in (3.1) was replaced by W_{comp} in (3.3), we replace W_n by

$$W_n(F) := \frac{\mu}{2}\left[\frac{\operatorname{tr}(F^T L_n^{-1} F)}{(\det F)^{2/3}} - 3\right] + W_{\text{vol}}(\det F), \quad \det F > 0, \tag{3.5}$$

where we have used the fact that $\det F_n = \det F$. We always assume that W_{vol} satisfies (3.4).

We work with the compressible model for nematic elastomers given by the minimum over n of the compressible densities W_n in (3.5):

$$W(F) := \min_{n\in\mathbb{S}^2} \frac{\mu}{2}\left[\frac{\operatorname{tr}(F^T L_n^{-1} F)}{(\det F)^{2/3}} - 3\right] + W_{\text{vol}}(\det F), \quad \det F > 0.$$

A straightforward computation (cf. [14]) gives

$$\min_{n\in\mathbb{S}^2} \operatorname{tr}(F^T L_n^{-1} F) = \left(\frac{\lambda_1(F)}{a^{-1/6}}\right)^2 + \left(\frac{\lambda_2(F)}{a^{-1/6}}\right)^2 + \left(\frac{\lambda_3(F)}{a^{1/3}}\right)^2, \tag{3.6}$$

where $0 < \lambda_1(F) \le \lambda_2(F) \le \lambda_3(F)$ are the ordered singular values of F. It is easy to check that $W(F) \ge 0$ and $W(F) = 0$ if and only if F belongs to the set

$$\mathcal{U} := \left\{ L_n^{\frac{1}{2}} R \,:\, n\in\mathbb{S}^2,\ R\in SO(3)\right\}.$$

Remark 3.1. In the sequel, we may equivalently consider wells of the form

$$\tilde{\mathcal{U}} := \left\{ R L_n^{\frac{1}{2}} \,:\, n\in\mathbb{S}^2,\ R\in SO(3)\right\}.$$

Indeed, we have that $\mathcal{U} = \tilde{\mathcal{U}}$, because $RL_n^{\frac{1}{2}} = L_{Rn}^{\frac{1}{2}} R$ for any $n\in\mathbb{S}^2$ and any $R\in SO(3)$.

3.1. **The small-strain regime and its rigorous justification.** We consider the small-strain regime $a = (1+\varepsilon)^3$, with $\varepsilon \ll 1$. In this case, we write

$$L_{n,\varepsilon} := (1+\varepsilon)^2 n \otimes n + (1+\varepsilon)^{-1}(I - n \otimes n), \tag{3.7}$$

and using (3.6),

$$W_\varepsilon(F) := \min_{n\in\mathbb{S}^2} \frac{\mu}{2}\left[\frac{\operatorname{tr}(F^T L_{n,\varepsilon}^{-1} F)}{(\det F)^{2/3}} - 3\right] + W_{\mathrm{vol}}(\det F) \tag{3.8}$$

$$= \frac{\mu}{2}\left\{\frac{1}{(\det F)^{2/3}}\left[\left(\frac{\lambda_1(F)}{(1+\varepsilon)^{-\frac12}}\right)^2 + \left(\frac{\lambda_2(F)}{(1+\varepsilon)^{-\frac12}}\right)^2 + \left(\frac{\lambda_3(F)}{1+\varepsilon}\right)^2 - 3\right]\right\} + W_{\mathrm{vol}}(\det F) \tag{3.9}$$

for all F with $\det F > 0$. The set of wells for W_ε is

$$\mathcal{U}_\varepsilon := \left\{ R L_{n,\varepsilon}^{\frac12} \,:\, n \in \mathbb{S}^2,\ R \in SO(3)\right\}. \tag{3.10}$$

Remark 3.2. From (3.7), we can write $L_{n,\varepsilon} = I + 2\varepsilon U_n + o(\varepsilon)$, where

$$U_n := \frac{1}{2}(3n \otimes n - I) \tag{3.11}$$

is traceless. This definition of U_n will be useful later on because we will use the equivalence $(I+\varepsilon E)^2 - L_{n,\varepsilon} = 2\varepsilon(E - U_n) + o(\varepsilon)$ to deduce the expression of the limiting small-strain energy density. We note that using $a = (1+\varepsilon)^\alpha$, $\alpha \in \mathbb{R}$, would also be a valid small-strain regime. In this case the first-order expansion of $L_{n,\varepsilon}$ would be the same as before but with $\frac{\alpha}{6}(3n\otimes n - I)$ in place of (3.11). The power $\alpha = 3$ is chosen only for notational convenience.

The results of Section 2 can be applied with $p = 3/2$ (see Lemma 3.2) to the model we have presented so far and in particular we can deduce Theorem 3.1 below. Here, Ω is a bounded and Lipschitz domain of $\mathbb{R}^3$ and, as in Section 2, $h \in W^{1,\infty}(\Omega,\mathbb{R}^3)$ represents the boundary data and $\mathscr{L} : W^{1,\frac32}(\Omega,\mathbb{R}^3) \to \mathbb{R}$, continuous and linear, represents the applied loads. The set $W_h^{1,\frac32}$ is defined as in (2.3) with $p = 3/2$.

Theorem 3.1. *Consider the family of energy densities given by (3.8). Set*

$$m_\varepsilon := \inf_{u\in W_h^{1,\frac32}} \left\{\frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(I + \varepsilon\nabla u)dx - \mathscr{L}(u)\right\},$$

and suppose that $\{u_\varepsilon\}$ *is a sequence such that*

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(I+\varepsilon\nabla u_\varepsilon)dx - \mathscr{L}(u_\varepsilon)\right\} = \lim_{\varepsilon\to0} m_\varepsilon.$$

Then, up to a subsequence, $u_\varepsilon \rightharpoonup u$ *in* $W^{1,\frac32}(\Omega,\mathbb{R}^3)$*, where* u *is a solution to the minimum problem*

$$m := \min_{u\in W_h^{1,2}}\left\{\int_\Omega f^{qc}(\nabla u)dx - \mathscr{L}(u)\right\},$$

with

$$f(F) := \mu \min_{n\in\mathbb{S}^2} |\operatorname{sym} F - U_n|^2 + \frac{\lambda}{2}\operatorname{tr}^2 F, \qquad \lambda := W''_{\mathrm{vol}}(1) - \frac{2}{3}\mu, \tag{3.12}$$

for every $F \in \mathbb{R}^3$*. Moreover,* $m_\varepsilon \to m$*.*

The choice of notation μ and λ for the constants in (3.12) is motivated by the theory of isotropic linear elasticity where μ and λ correspond to the shear and bulk modulus, respectively.

We remark that an explicit expression for the quasiconvexification f^{qc} of f, as given in (3.13) below, is due to [7]. It turns out that, while the expression of f involves the distance of $\operatorname{sym} F$ to the set of matrices

$$\{U_n : n \in \mathbb{S}^2\} = \Big\{U \in \mathbb{R}^{3\times 3}_{\rm sym} : \{\text{eigenvalues of } U\} = \{-1/2, -1/2, 1\}\Big\}$$

(where U_n is defined in (3.11)), the expression of f^{qc} involves the distance of $\operatorname{sym} F$ to the set

$$\mathscr{Q} := \Big\{U \in \mathbb{R}^{3\times 3}_{\rm sym} : \operatorname{tr} U = 0 \text{ and } \{\text{eigenvalues of } U\} \subseteq [-1/2, 1]\Big\}.$$

More precisely,

$$f^{qc}(F) = \mu \min_{U \in \mathscr{Q}} |\operatorname{sym} F - U|^2 + \frac{\lambda}{2}\operatorname{tr}^2 F, \tag{3.13}$$

for every $F \in \mathbb{R}^{3\times 3}$.

Theorem 3.1 is a straightforward application of Theorem 2.1 once we establish that the family of energies $\{W_\varepsilon\}$ given by (3.8) satisfies the hypotheses. Essentially, we have to verify that the growth condition (2.1) with $p = 3/2$ and $\mathcal{U}_\varepsilon$ given by (3.10) is satisfied (see Lemma 3.2). Also, we have to show that f as in (3.12) satisfies $f(F) = V(\operatorname{sym} F)$, where $W_\varepsilon(I + \varepsilon\cdot)/\varepsilon^2 \to V$, as $\varepsilon \to 0$, uniformly on compact subsets of $\mathbb{R}^{3\times 3}_{\rm sym}$ (see Lemma 3.3). The fact that $0 \le f(F) \le c(1 + |F|^2)$ is a direct consequence of (3.12).

In establishing estimates, it is useful to define the functions

$$\widetilde{W}_{n,\varepsilon}(B) := \frac{\mu}{2}\left(\frac{\operatorname{tr}(BL^{-1}_{n,\varepsilon})}{(\det B)^{1/3}} - 3\right) + W_{\rm vol}(\sqrt{\det B}\,), \quad \widetilde{W}_\varepsilon(B) := \min_{n\in\mathbb{S}^2} \widetilde{W}_{n,\varepsilon}(B), \tag{3.14}$$

for every positive definite, symmetric matrix B, to replace $W_{n,\varepsilon}$ and W_ε. Indeed,

$$W_{n,\varepsilon}(F) = \widetilde{W}_{n,\varepsilon}(FF^T), \qquad W_\varepsilon(F) = \widetilde{W}_\varepsilon(FF^T), \tag{3.15}$$

for every $F \in \mathbb{R}^{3\times 3}$ with $\det F > 0$.

Lemma 3.2. *Let W_ε be defined as in* (3.8)*, and $\mathcal{U}_\varepsilon$ as in* (3.10)*. Then the following holds:*

(i) $W_\varepsilon(F) \ge c\,\operatorname{dist}^2(F, \mathcal{U}_\varepsilon)$ *for F near $\mathcal{U}_\varepsilon$,*
(ii) $W_\varepsilon(F) \ge c\,\operatorname{dist}^{3/2}(F, \mathcal{U}_\varepsilon)$ *for F far from $\mathcal{U}_\varepsilon$.*

Note that the above lemma implies that, up to a multiplicative constant, $W_\varepsilon(F)$ is bounded below by $g_{\frac{3}{2}}(\operatorname{dist}(F, \mathcal{U}_\varepsilon))$, where $g_{\frac{3}{2}}$ is the function given by (2.2) with $p = 3/2$.

Proof. Note that $\widetilde{W}_{n,\varepsilon}(B)$ is minimized at the level 0 by $B = L_{n,\varepsilon}$, so that Taylor expansion gives

$$\widetilde{W}_{n,\varepsilon}(B) = \frac{1}{2} D^2\widetilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[B - L_{n,\varepsilon}]^2 + o\left(|B - L_{n,\varepsilon}|^2\right). \tag{3.16}$$

A direct computation yields

$$D^2\widetilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[H]^2 = \left(\frac{1}{4}W''_{\rm vol}(1) - \frac{\mu}{6}\right)\operatorname{tr}^2(HL^{-1}_{n,\varepsilon}) + \frac{\mu}{2}\operatorname{tr}(HL^{-1}_{n,\varepsilon})^2, \tag{3.17}$$

for every $H \in \mathbb{R}^{d\times d}_{\rm sym}$. Hence, if $\frac{1}{4}W''_{\rm vol}(1) - \frac{\mu}{6} \geq 0$ then $D^2\widetilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[H]^2 \geq \frac{\mu}{2}\mathrm{tr}(HL^{-1}_{n,\varepsilon})^2$. On the other hand, if $\frac{1}{4}W''_{\rm vol}(1) - \frac{\mu}{6} < 0$ then

$$D^2\widetilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[H]^2 \geq \frac{3}{4}W''_{\rm vol}(1)\mathrm{tr}(HL^{-1}_{n,\varepsilon})^2.$$

This is due to the fact that $\mathrm{tr}(HL^{-1}_{n,\varepsilon}) = \mathrm{tr}(L^{-\frac{1}{2}}_{n,\varepsilon}HL^{-\frac{1}{2}}_{n,\varepsilon})$ and that $(\mathrm{tr}A)^2 \leq 3\,\mathrm{tr}A^2$ for any $A \in \mathbb{R}^{3\times 3}_{\rm sym}$. Thus,

$$D^2\widetilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[H]^2 \geq \min\left\{\frac{\mu}{2}, \frac{3}{4}W''_{\rm vol}(1)\right\}\mathrm{tr}(HL^{-1}_{n,\varepsilon})^2. \tag{3.18}$$

Now, since $\mathbb{S}^2$ is compact, one can show that

$$\mathrm{tr}(HL^{-1}_{n,\varepsilon})^2 \geq \frac{1}{2}|H|^2, \tag{3.19}$$

for every $n \in \mathbb{S}^2$, $H \in \mathbb{R}^{d\times d}_{\rm sym}$, and for all ε small enough (independently of n and H). Then, from (3.16), (3.18) and (3.19), we obtain

$$\widetilde{W}_{n,\varepsilon}(B) \geq \frac{1}{4}\min\left\{\frac{\mu}{2}, \frac{3}{4}W''_{\rm vol}(1)\right\}|B - L_{n,\varepsilon}|^2 + o\left(|B - L_{n,\varepsilon}|^2\right).$$

Thus, provided $|B - L_{n,\varepsilon}|$ is small, we have that

$$\begin{aligned} W_{n,\varepsilon}(F) = \widetilde{W}_{n,\varepsilon}(FF^T) &\geq c\left|FF^T - L_{n,\varepsilon}\right|^2 \geq c\left|\sqrt{FF^T} - L^{\frac{1}{2}}_{n,\varepsilon}\right|^2 \\ &\geq c\min_{Q\in SO(3)}\left|F - L^{\frac{1}{2}}_{n,\varepsilon}Q\right|^2 \geq c\min_{n\in\mathbb{S}^2}\min_{Q\in SO(3)}\left|F - L^{\frac{1}{2}}_{n,\varepsilon}Q\right|^2 \\ &= c\,\mathrm{dist}^2(F,\mathcal{U}_\varepsilon), \end{aligned} \tag{3.20}$$

where we have also used the fact that $|\sqrt{F} - \sqrt{G}| \leq c|F - G|$ for any two positive definite matrices F and G sufficiently close to the identity. Since c in (3.20) is independent of n, we then have $\min_{n\in\mathbb{S}^2} W_{n,\varepsilon}(F) \geq c\,\mathrm{dist}^2(F,\mathcal{U}_\varepsilon)$, whenever $\mathrm{dist}(F,\mathcal{U}_\varepsilon)$ and ε are sufficiently small. This establishes (i).

Without loss of generality, we can assume $\varepsilon \leq 1$ so that $1 \leq (1+\varepsilon)^2 \leq 4$. Hence,

$$\left(\frac{\lambda_1}{(1+\varepsilon)^{-\frac{1}{2}}}\right)^2 + \left(\frac{\lambda_2}{(1+\varepsilon)^{-\frac{1}{2}}}\right)^2 + \left(\frac{\lambda_3}{1+\varepsilon}\right)^2 \geq \lambda_1^2 + \lambda_2^2 + \frac{\lambda_3^2}{4},$$

and from (3.9)

$$W_\varepsilon(F) \geq \frac{\mu}{2}\left(\frac{|F|^2}{4(\det F)^{2/3}} - 3\right) + W_{\rm vol}(\det F). \tag{3.21}$$

There are two cases. Either $\det F < M$ or $\det F \geq M$, where M is the constant in (3.4). In the case $\det F < M$, from (3.21) we obtain

$$W_\varepsilon(F) \geq c_1|F|^2 - c_2 \geq c\,\mathrm{dist}^2(F,\mathcal{U}_\varepsilon)$$

for $|F| \gg 1$, since $W_{\rm vol} \geq 0$. Now, if $\det F \geq M$, we know from (3.4) that $W_{\rm vol}(\det F) \geq k(\det F)^2$. Hence, it follows from (3.21) that

$$W_\varepsilon(F) \geq \min\left\{\frac{\mu}{8}, k\right\}\left(\frac{|F|^2}{(\det F)^{2/3}} + (\det F)^2\right) - \frac{3\mu}{2}.$$

Applying Young's inequality $xy \leq \frac{1}{r}x^r + \frac{1}{q}y^q$ with $x = (\det F)^{-1/2}|F|^{3/2}$, $y = (\det F)^{1/2}$, and $r = 4/3$, $q = 4$, we have

$$W_\varepsilon(F) \geq \frac{4}{3}\min\left\{\frac{\mu}{8}, k\right\}\left(\frac{3}{4}x^{4/3} + \frac{1}{4}y^4\right) - \frac{3\mu}{2} \geq \frac{4}{3}\min\left\{\frac{\mu}{8}, k\right\}|F|^{3/2} - \frac{3\mu}{2}.$$

Thus, for $|F| \gg 1$, $W_\varepsilon(F) \geq c\,\mathrm{dist}^{3/2}(F,\mathcal{U}_\varepsilon)$. □

Lemma 3.3. *Let U_n be defined as in* (3.11). *For $E \in \mathbb{R}^{3\times 3}_{\rm sym}$, we define*

$$V(E) = \mu \min_{n\in\mathbb{S}^2} |E - U_n|^2 + \frac{\lambda}{2}\mathrm{tr}^2 E, \qquad \lambda = W''_{\rm vol}(1) - \frac{2}{3}\mu.$$

Then

$$V(E) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} W_\varepsilon(I + \varepsilon E), \tag{3.22}$$

where W_ε is defined in (3.8), *and the limit is uniform on compact subsets of $\mathbb{R}^{3\times 3}_{\rm sym}$.*

Proof. For every $E \in \mathbb{R}^{3\times 3}_{\rm sym}$, let us define

$$V_n(E) := 2D^2\widetilde{W}_{n,0}(I)[E - U_n]^2, \qquad \tilde{V}(E) := \min_{n\in\mathbb{S}^2} V_n(E),$$

where $\widetilde{W}_{n,0}$ is given by (3.14) with $\varepsilon = 0$. Note that from (3.17) with $\varepsilon = 0$ we have that

$$D^2\widetilde{W}_{n,0}(I)[H]^2 = \left(\frac{1}{4}W''_{\rm vol}(1) - \frac{\mu}{6}\right)\mathrm{tr}^2 H + \frac{\mu}{2}|H|^2,$$

for every $H \in \mathbb{R}^{3\times 3}_{\rm sym}$, so that

$$\begin{aligned}\tilde{V}(E) &= \min_{n\in\mathbb{S}^2}\left(\frac{1}{2}W''_{\rm vol}(1) - \frac{\mu}{3}\right)\mathrm{tr}^2(E - U_n) + \mu|E - U_n|^2 \\ &= \min_{n\in\mathbb{S}^2} \mu|E - U_n|^2 + \frac{\lambda}{2}\mathrm{tr}^2 E,\end{aligned}$$

with $\lambda = W''_{\rm vol}(1) - \frac{2}{3}\mu$, in view of the fact that $\mathrm{tr}\, U_n = 0$. Thus, $\tilde{V} = V$ where V is given by (3.22). Now, for $W_{n,\varepsilon}$ defined in (3.5) with $L_{n,\varepsilon}$ in place of L_n, let us introduce for every $E \in \mathbb{R}^{3\times 3}_{\rm sym}$

$$q_{n,\varepsilon}(E) := \frac{1}{\varepsilon^2}W_{n,\varepsilon}(I + \varepsilon E), \quad q_\varepsilon(E) := \min_{n\in\mathbb{S}^2} q_{n,\varepsilon}(E) = \frac{1}{\varepsilon^2}W_\varepsilon(I + \varepsilon E).$$

To prove the lemma, we show that $q_{\varepsilon_j} \to \tilde{V}$, uniformly on compact subsets of $\mathbb{R}^{d\times d}_{\rm sym}$, for every vanishing sequence $\{\varepsilon_j\}$.

Given a compact $K \subset \mathbb{R}^{3\times 3}_{\rm sym}$, we prove that $\sup_K(q_{\varepsilon_j} - \tilde{V}) \to 0$ and $\inf_K(q_{\varepsilon_j} - \tilde{V}) \to 0$, so that $\sup_K |q_\varepsilon - \tilde{V}| = \max\{\sup_K(q_{\varepsilon_j} - \tilde{V}), -\inf_K(q_{\varepsilon_j} - \tilde{V})\} \to 0$. Note that q_{ε_j} and $\tilde{V}$ are both continuous so

$$\sup_K(q_{\varepsilon_j} - \tilde{V}) = q_{\varepsilon_j}(E_{\varepsilon_j}) - \tilde{V}(E_{\varepsilon_j})$$

for some $E_{\varepsilon_j} \in K$. Up to subsequences, we have that $E_{\varepsilon_j} \to E$, as $j \to \infty$, for some $E \in K$. For any $n \in \mathbb{S}^2$,

$$\sup_K(q_{\varepsilon_j} - \tilde{V}) \le q_{n,\varepsilon_j}(E_{\varepsilon_j}) - \tilde{V}(E_{\varepsilon_j}) \le \sup_K |q_{n,\varepsilon_j} - V_n| + V_n(E_{\varepsilon_j}) - \tilde{V}(E_{\varepsilon_j}). \tag{3.23}$$

By Lemma 3.4 below, $q_{n,\varepsilon} \to V_n$ uniformly on K, as $\varepsilon \to 0$, for every $n \in \mathbb{S}^2$. Therefore, by continuity of V_n, we obtain from (3.23) that

$$\limsup_{j\to\infty} \sup_K(q_{\varepsilon_j} - \tilde{V}) \le V_n(E) - \tilde{V}(E), \quad \text{for every } n \in \mathbb{S}^2.$$

Taking the minimum over $n \in \mathbb{S}^2$ implies

$$\limsup_{j\to\infty} \sup_K(q_{\varepsilon_j} - \tilde{V}) \le \min_{n\in\mathbb{S}^2} V_n(E) - \tilde{V}(E) = 0. \tag{3.24}$$

On the other hand,

$$\sup_K(q_{\varepsilon_j} - \tilde{V}) = q_{\varepsilon_j}(E_{\varepsilon_j}) - \tilde{V}(E_{\varepsilon_j}) = q_{n_j,\varepsilon_j}(E_{\varepsilon_j}) - \tilde{V}(E_{\varepsilon_j}),$$

where $n_j \in \mathbb{S}^2$ attains the minimum. Up to taking a further subsequence, we may assume that $n_j \to \hat{n}$ as $j \to \infty$. Now

$$|q_{n_j,\varepsilon_j}(E_{\varepsilon_j}) - V_{\hat{n}}(E)| \leq \sup_K |q_{n_j,\varepsilon_j} - V_{\hat{n}}| + |V_{\hat{n}}(E_{\varepsilon_j}) - V_{\hat{n}}(E)|,$$

and $\sup_K |q_{n_j,\varepsilon_j} - V_{\hat{n}}| \to 0$ by elementary computations. Thus, $q_{n_j,\varepsilon_j}(E_{\varepsilon_j}) \to V_{\hat{n}}(E)$ and

$$\liminf_{j\to\infty} \sup_K (q_{\varepsilon_j} - \tilde{V}) = V_{\hat{n}}(E) - \tilde{V}(E) \geq \min_{n\in\mathbb{S}^2} V_n(E) - \tilde{V}(E) = 0. \tag{3.25}$$

Together, (3.24) and (3.25) imply

$$\lim_{j\to\infty} \sup_K (q_{\varepsilon_j} - \tilde{V}) = 0. \tag{3.26}$$

Establishing (3.26) with $\sup_K$ replaced by $\inf_K$ is very similar. Indeed, let E_{ε_j} be such that

$$\inf_K (q_{\varepsilon_j} - \tilde{V}) = q_{\varepsilon_j}(E_{\varepsilon_j}) - \tilde{V}(E_{\varepsilon_j}),$$

to obtain an $E \in K$ and $\{n_j\} \subset \mathbb{S}^2$ such that $E_{\varepsilon_j} \to E$, $n_j \to \hat{n}$, and $q_{\varepsilon_j}(E_{\varepsilon_j}) = q_{n_j,\varepsilon_j}(E_{\varepsilon_j})$. For any $n \in \mathbb{S}^2$, $\inf_K(q_{\varepsilon_j} - \tilde{V}) \leq q_{n,\varepsilon_j}(E_{\varepsilon_j}) - \tilde{V}(E_{\varepsilon_j})$ and, just as before, $q_{n,\varepsilon_j}(E_{\varepsilon_j}) \to V_n(E)$. Thus

$$\limsup_{j\to\infty} \inf_K (q_{\varepsilon_j} - \tilde{V}) \leq V_n(E) - \tilde{V}(E), \quad \text{for every } n \in \mathbb{S}^2,$$

and in turn

$$\limsup_{j\to\infty} \inf_K (q_{\varepsilon_j} - \tilde{V}) \leq \min_{n\in\mathbb{S}^2} V_n(E) - \tilde{V}(E) = 0. \tag{3.27}$$

On the other hand, as before $q_{n_j,\varepsilon_j}(E_{\varepsilon_j}) \to V_{\hat{n}}(E)$ so that

$$\begin{aligned}\liminf_{j\to\infty} \inf_K (q_{\varepsilon_j} - \tilde{V}) &= \liminf_{j\to\infty} \Big(q_{n_j,\varepsilon_j}(E_{\varepsilon_j}) - \tilde{V}(E_{\varepsilon_j})\Big) \\ &= V_{\hat{n}}(E) - \tilde{V}(E) \geq \min_{n\in\mathbb{S}^2} V_{\hat{n}}(E) - \tilde{V}(E) = 0.\end{aligned}$$

This, together with (3.27), implies

$$\lim_{j\to\infty} \inf_K (q_{\varepsilon_j} - V) = 0. \tag{3.28}$$

Equations (3.26) and (3.28) complete the proof.

□

Lemma 3.4. *For all $n \in \mathbb{S}^2$, $q_{n,\varepsilon} \longrightarrow V_n$, as $\varepsilon \to 0$, uniformly on compact subsets of $\mathbb{R}^{3\times3}_{\text{sym}}$.*

Proof. Let $K \subset \mathbb{R}^{3\times3}_{\text{sym}}$ be compact. Recall from (3.15) that for every $E \in \mathbb{R}^{3\times3}_{\text{sym}}$, $W_{n,\varepsilon}(I + \varepsilon E) = \widetilde{W}_{n,\varepsilon}((I + \varepsilon E)^2)$ and that $(I + \varepsilon E)^2 - L_{n,\varepsilon} = 2\varepsilon(E - U_n) + o(\varepsilon)$, from Remark 3.2. Thus, for every $E \in K$ we have by Taylor expansion that

$$\begin{aligned}q_{n,\varepsilon}(E) &= \frac{1}{2\varepsilon^2} D^2\widetilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[(I + \varepsilon E)^2 - L_{n,\varepsilon}]^2 + \frac{1}{\varepsilon^2} o(|(I + \varepsilon E)^2 - L_{n,\varepsilon}|^2) \\ &= 2D^2\widetilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[E - U_n + o(1)]^2 + o(1).\end{aligned}$$

Adding and subtracting $2D^2\widetilde{W}_{n,0}(L_{n,\varepsilon}))[E-U_n+o(1)]^2$ and taking the supremum over $E \in K$ gives

$$\begin{aligned}\sup_K |q_{n,\varepsilon} - V_n| &\le 2 \sup_{E\in K} \Big\{ \Big| (D^2\widetilde{W}_{n,\varepsilon}(L_{n,\varepsilon}) - D^2\widetilde{W}_{n,0}(L_{n,\varepsilon}))[E-U_n+o(1)]^2 \Big| \\ &\quad + \Big| D^2\widetilde{W}_{n,0}(L_{n,\varepsilon})[E-U_n+o(1)]^2 - \frac{V_n(E)}{2} + o(1) \Big| \Big\} \\ &\le C \sup_{M\in K} \Big| D^2\widetilde{W}_{n,\varepsilon}(M) - D^2\widetilde{W}_{n,0}(M) \Big| \\ &\qquad + C \Big| D^2\widetilde{W}_{n,0}(L_{n,\varepsilon}) - D^2\widetilde{W}_{n,0}(I) \Big| + o(1),\end{aligned}$$

where in the last inequality we have used the definition of V_n. By elementary computations one can verify that the summands on the right side of the last inequality tend to 0 as $\varepsilon \to 0$. □

4. Proofs of the main results

For the sequential characterization of Γ-convergence, as well as to prove that almost minimizers of $\mathscr{E}_\varepsilon$ converge to minimizers of $\overline{\mathscr{E}}$, we need to establish that the functionals $\mathscr{E}_\varepsilon$ are equicoercive; this is the content of Proposition 2.2. Before proving it, we collect some useful properties of the function g_p defined in (2.2).

Lemma 4.1. *The function g_p satisfies the following:*

(i) *g_p is convex;*
(ii) *$g_p(s+t) \le C(g_p(s)+t^2)$ for all $s,t \ge 0$, where $C > 0$ is a constant depending only on p;*
(iii) *for each $K > 0$ there exists a constant $C > 0$ depending on K and p such that*

$$\begin{aligned} t^2 &\le C g_p(t), \qquad 0 \le t \le K \\ t^p &\le C g_p(t), \qquad t \ge K. \end{aligned}$$

The proof of this lemma is elementary and left to the reader.

Proof of Proposition 2.2. We may assume that $u \in W_h^{1,p}$, otherwise the result follows trivially. Supposing that the bound

$$\int_\Omega g_p(|\varepsilon \nabla u(x)|)dx \le C\varepsilon^2(1+\mathscr{E}_\varepsilon(u)) \tag{4.1}$$

holds, Proposition 2.2 can be established by estimating $\|\varepsilon\nabla u\|_p^p$. Indeed, by Hölder's inequality and the definition of g_p in (2.2),

$$\int_{\{x\in\Omega:|\varepsilon\nabla u(x)|\le 1\}} |\varepsilon\nabla u(x)|^p dx \le C \left(\int_{\{x\in\Omega:|\varepsilon\nabla u(x)|\le 1\}} g_p(|\varepsilon\nabla u(x)|)dx \right)^{\frac{p}{2}}.$$

Thus, using (4.1) and the fact that $t^{p/2} \le t+1$ for every $t \ge 0$, we obtain

$$\int_{\{x\in\Omega:|\varepsilon\nabla u(x)|\le 1\}} |\varepsilon\nabla u(x)|^p dx \le C\varepsilon^p \left(1+\mathscr{E}_\varepsilon(u)\right)^{\frac{p}{2}} \le 2C\varepsilon^p(1+\mathscr{E}_\varepsilon(u)), \tag{4.2}$$

because $\mathscr{E}_\varepsilon(u) \ge 0$. On the other hand, by Lemma 4.1 (iii) and (4.1),

$$\begin{aligned}\int_{\{x\in\Omega:|\varepsilon\nabla u(x)|>1\}} |\varepsilon\nabla u(x)|^p dx &\le C \int_{\{x\in\Omega:|\varepsilon\nabla u(x)|>1\}} g_p(|\varepsilon\nabla u(x)|) \\ &\le C\varepsilon^2(1+\mathscr{E}_\varepsilon(u)) \le C\varepsilon^p(1+\mathscr{E}_\varepsilon(u)), \quad (4.3)\end{aligned}$$

since $p \le 2$ and ε is small. The compactness result now follows by (4.2) and (4.3). Hence, to complete the proof, we need only establish (4.1).

Note that, by the coercivity condition (2.1),

$$\int_\Omega g_p(\mathrm{dist}(I+\varepsilon\nabla u(x),\mathcal{U}_\varepsilon))\,dx \le C\varepsilon^2\mathscr{E}_\varepsilon(u)$$

and, since $\mathcal{U}_\varepsilon$ is compact, there exist $R_\varepsilon(x)U_\varepsilon(x)\in\mathcal{U}_\varepsilon$ such that the distance is achieved for a.e. $x\in\Omega$, i.e.

$$\int_\Omega g_p(|I+\varepsilon\nabla u(x)-R_\varepsilon(x)U_\varepsilon(x)|)\,dx \le C\varepsilon^2\mathscr{E}_\varepsilon(u). \tag{4.4}$$

In order to apply the rigidity result of Friesecke, James and Müller [15] we need a lower bound for $g_p(|1+\varepsilon\nabla u(x)-R_\varepsilon(x)U_\varepsilon(x)|)$ in terms of the distance of $I+\varepsilon\nabla u(x)$ to $SO(d)$. But since g_p is increasing, by Lemma 4.1 (ii) we infer that for $v(x)=x+\varepsilon u(x)$,

$$\begin{aligned} g_p(\mathrm{dist}(\nabla v(x),SO(d))) &\le g_p(|\nabla v(x)-R_\varepsilon(x)|)\\ &\le g_p(|\nabla v(x)-R_\varepsilon(x)U_\varepsilon(x)|+|R_\varepsilon(x)U_\varepsilon(x)-R_\varepsilon(x)|)\\ &\le c\{g_p(|\nabla v(x)-R_\varepsilon(x)U_\varepsilon(x)|)+|U_\varepsilon(x)-I|^2\} \qquad (4.5)\end{aligned}$$

But $U_\varepsilon(x)=I+\varepsilon U(x)+o(\varepsilon)$ for some $U(x)$ in the compact set $\mathcal{M}$ implies that

$$|U_\varepsilon(x)-I|=\varepsilon|U(x)-o(1)|\le c\,\varepsilon$$

so that, by (4.4) and (4.5),

$$\int_\Omega g_p(\mathrm{dist}(\nabla v(x),SO(d)))\,dx \le c\,\varepsilon^2(\mathscr{E}_\varepsilon(u)+1). \tag{4.6}$$

Now we may apply the modified rigidity result of [15] (cf. [4, Lemma 3.1]) to get the existence of an x-independent $R_\varepsilon\in SO(d)$ such that, in conjunction with (4.6),

$$\int_\Omega g_p(|\nabla v(x)-R_\varepsilon|)\,dx \le c\int_\Omega g_p(\mathrm{dist}(\nabla v(x),SO(d)))\,dx \le c\,\varepsilon^2(\mathscr{E}_\varepsilon(u)+1). \tag{4.7}$$

Also, by [4, Lemma 3.3], we infer that

$$|R_\varepsilon-I|^2\le c\,\varepsilon^2\left[\mathscr{E}_\varepsilon(u)+\left(\int_{\partial_D\Omega}|h|d\mathscr{H}^{n-1}\right)^2\right]\le c\,\varepsilon^2(\mathscr{E}_\varepsilon(u)+1), \tag{4.8}$$

where c now depends on h and $\partial_D\Omega$ as well. To complete the proof, note that as before, g_p being increasing, a further application of Lemma 4.1 (ii) shows that

$$\begin{aligned}\int_\Omega g_p(|\varepsilon\nabla u(x)|)dx &\le \int_\Omega g_p(|I+\varepsilon\nabla u(x)-R_\varepsilon|+|R_\varepsilon-I|)\\ &\le C\int_\Omega g_p(|I+\varepsilon\nabla u(x)-R_\varepsilon|)+|R_\varepsilon-I|^2dx\\ &\le C\varepsilon^2(I+\mathscr{E}_\varepsilon(u))\end{aligned}$$

by (4.7) and (4.8), establishing (4.1). □

Before proving Theorem 2.3, we state two auxiliary key results which are used in the proof. For the proof of Lemma 4.2, we refer the reader to [4]; Lemma 4.3 is due to J. Kristensen [20]. In what follows, given a set $B\subset\mathbb{R}^d$, we denote by 1_B its characteristic function.

Lemma 4.2. *Let $\varepsilon_j\to 0$, as $j\to\infty$. Suppose that $\{\mathscr{E}_{\varepsilon_j}(u_j)\}$ is bounded for some sequence $\{u_j\}\subset W^{1,p}(\Omega,\mathbb{R}^d)$ and that, in view of compactness, $u_j\rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^d)$. For each j, define the sets*

$$B_j:=\left\{x\in\Omega:|\nabla u_j(x)|\le\frac{1}{\sqrt{\varepsilon_j}}\right\}. \tag{4.9}$$

Then, the following holds:

(i) $|B_j^c| \to 0$ *and* $1_{B_j^c}\nabla u_j \to 0$ *in* $L^q(\Omega,\mathbb{R}^{d\times d})$, *for all* $1 \le q < p$;
(ii) $\nabla u \in L^2(\Omega,\mathbb{R}^{d\times d})$ *and* $1_{B_j}\nabla u_j \rightharpoonup \nabla u$ *in* $L^2(\Omega,\mathbb{R}^{d\times d})$.

Lemma 4.3 (Proposition 1.10, [20])**.** *Let* $g : \mathbb{R}^{d\times d} \to \mathbb{R}$ *be a quasiconvex function such that for some* c_1, $c_2 > 0$ *and* $p > 1$

$$c_1|F|^p - c_2 \le g(F) \le c_2(1 + |F|^p), \quad \textit{for all } F \in \mathbb{R}^{d\times d}.$$

Then there exists a nondecreasing sequence of quasiconvex functions $\psi_k : \mathbb{R}^{d\times d} \to \mathbb{R}$, *bounded above by* g, *such that* $\{\psi_k\}$ *converges to* g *pointwise and*

$$\psi_k(F) = a_k|F| + b_k, \quad \textit{for all } |F| \ge r_k,$$

for some a_k, $r_k > 0$ *and* $b_k \in \mathbb{R}$.

Proof of Theorem 2.3. It suffices to show that, fixing a vanishing sequence $\{\varepsilon_j\}$, $\{\mathscr{E}_{\varepsilon_j}\}$ Γ-converges to $\overline{\mathscr{E}}$.

To establish the Γ-lim inf inequality, we follow the lines of the proof of the Γ-lim inf inequality in [23] with some modifications.

Let $u_j \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^d)$ and assume that $\liminf \mathscr{E}_{\varepsilon_j}(u_j) < \infty$, as otherwise, the result follows trivially. In particular, we may assume that, up to a subsequence, $\mathscr{E}_{\varepsilon_j}(u_j) \le c < \infty$, so that, in particular, u_j, $u \in W_h^{1,p}$, and

$$\mathscr{E}_{\varepsilon_j}(u_j) \ge \int_{B_j} \frac{1}{\varepsilon_j^2} W_{\varepsilon_j}\left(\sqrt{(I+\varepsilon_j\nabla u_j)^T(I+\varepsilon_j\nabla u_j)}\right)dx,$$

in view of frame-indifference. Note that the above integral is taken over the set B_j, defined in (4.9), where we may assume that the determinant of $I+\varepsilon_j\nabla u_j$ is bounded away from zero and apply the polar decomposition. It is useful to introduce the function $\zeta : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\rm sym}$ defined as

$$\zeta(F) := \sqrt{(I+F)^T(I+F)} - I - \operatorname{sym} F, \tag{4.10}$$

which satisfies

$$\zeta(F) \le c\min\{|F|, |F|^2\}, \qquad \text{for every } F \in \mathbb{R}^{d\times d}, \tag{4.11}$$

and to use the notation (2.4)-(2.5) to write

$$\begin{aligned} W_{\varepsilon_j}\left(\sqrt{(I+\varepsilon_jF)^T(I+\varepsilon_jF)}\right) &= W_{\varepsilon_j}\left(I + \varepsilon_j\left(\operatorname{sym}F + \frac{\zeta(\varepsilon_jF)}{\varepsilon_j}\right)\right) \\ &= \varepsilon_j^2 f_{\varepsilon_j}\left(F + \frac{\zeta(\varepsilon_jF)}{\varepsilon_j}\right). \end{aligned}$$

Note also that we may exploit the boundedness of the sequence $\{1_{B_j}\nabla u_j\}$ in $L^2(\Omega,\mathbb{R}^{d\times d})$ given by Lemma 4.2 to get that

$$\mathscr{E}_{\varepsilon_j}(u_j) \ge \int_{B_j} f_{\varepsilon_j}\left(\nabla u_j + \frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right)dx. \tag{4.12}$$

Consider the function f which is the uniform limit of the sequence $\{f_{\varepsilon_j}\}$ on compact subsets of $\mathbb{R}^{d\times d}$. To employ the approximation result of Lemma 4.3 which requires quadratic growth from below, fix $\delta > 0$ arbitrarily and consider the function

$$g(F) := f^{qc}(F) + \delta|F|^2.$$

Note that g is quasiconvex and satisfies

$$\delta|F|^2 \le g(F) \le c(1 + |F|^2), \qquad \text{for all } F \in \mathbb{R}^{d\times d}.$$

Then, by Lemma 4.3, there exists a nondecreasing sequence of quasiconvex functions $\psi_k \le g$, converging to g pointwise and such that $\psi_k(F) = a_k|F| + b_k$ for all $|F| \ge r_k$, for some $a_k, r_k > 0$ and $b_k \in \mathbb{R}$. Now, observe that for every k there exists $\hat{j} = \hat{j}(\delta, k)$ such that

$$f_{\varepsilon_j}(F) + \delta|F|^2 \geq \psi_k(F) - \frac{1}{k}, \quad \text{for every } F \in \mathbb{R}^{d\times d},\ j \geq \hat{j}. \tag{4.13}$$

This is because for every k there exists $\hat{c} = \hat{c}(\delta, k) \geq r_k$ such that

$$\delta|F|^2 \geq a_k|F| + b_k - \frac{1}{k} = \psi_k(F) - \frac{1}{k}, \qquad \text{for every } |F| > \hat{c}.$$

Moreover, since $f_{\varepsilon_j} \to f$ uniformly on $\Omega_k := \{F \in \mathbb{R}^{d\times d} : |F| \leq \hat{c}\}$, there exists $\hat{j} = \hat{j}(\delta, k)$ such that

$$f_{\varepsilon_j}(F) + \delta|F|^2 \geq f(F) + \delta|F|^2 - \frac{1}{k} \geq g^{qc}(F) - \frac{1}{k} \geq \psi_k(F) - \frac{1}{k},$$

for every $F \in \Omega_k$ and all $j \geq \hat{j}$.

Using (4.12) and (4.13) we can then write

$$\mathscr{E}_{\varepsilon_j}(u_j) \geq \int_{B_j} \left\{ \psi_k\left(\nabla u_j + \frac{\zeta(\varepsilon_j \nabla u_j)}{\varepsilon_j}\right) - \frac{1}{k} - \delta\left|\nabla u_j + \frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right|^2 \right\} dx. \tag{4.14}$$

Focusing on the first term on the right-hand side of (4.14), note that

$$\begin{aligned}\int_{B_j} \psi_k\left(\nabla u_j + \frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right) dx \\ \geq \int_{B_j} \psi_k(\nabla u_j)\, dx - \int_{B_j} \left|\psi_k(\nabla u_j) - \psi_k\left(\nabla u_j + \frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right)\right| dx, \end{aligned}\tag{4.15}$$

and that the second summand on the right-hand side of (4.15) is bounded by

$$\int_{\{|\nabla u_j|\leq M\}} \left|\psi_k(\nabla u_j) - \psi_k\left(\nabla u_j + \frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right)\right| dx \\ + \int_{B_j\cap\{|\nabla u_j|>M\}} \left(|\psi_k(\nabla u_j)| + \left|\psi_k\left(\nabla u_j + \frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right)\right|\right) dx,$$

for a fixed $M > 0$. Therefore, on the one hand, since each ψ_k is quasiconvex and hence locally Lipschitz (see [10, Theorem 2.31]), we have that

$$\begin{aligned}\int_{\{|\nabla u_j|\leq M\}} \left|\psi_k(\nabla u_j) - \psi_k\left(\nabla u_j + \frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right)\right| dx \\ \leq c \int_{\{|\nabla u_j|\leq M\}} \left|\frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right| dx \\ \leq c\,\varepsilon_j \int_{\{|\nabla u_j|\leq M\}} |\nabla u_j|^2\, dx \leq c\,\varepsilon_j M^2, \end{aligned}\tag{4.16}$$

where in the second inequality we have used (4.11) and $c = c(k, M)$. On the other hand, since $\psi_k \leq \tilde{a}_k|\cdot| + \tilde{b}_k$ for suitable constants $\tilde{a}_k$, $\tilde{b}_k$, then

$$\begin{aligned}\int_{B_j\cap\{|\nabla u_j|>M\}} \left(|\psi_k(\nabla u_j)| + \left|\psi_k\left(\nabla u_j + \frac{\zeta(\varepsilon_j\nabla u_j)}{\varepsilon_j}\right)\right|\right) dx \\ \leq \int_{\{|\nabla u_j|>M\}} \left[(2+c)\tilde{a}_k|\nabla u_j| + 2\tilde{b}_k\right] dx, \end{aligned}\tag{4.17}$$

where we have also used (4.11). Now, by the equiintegrability of $\{|\nabla u_j|\}$, we can choose $M = M_k$ such that

$$\int_{\{|\nabla u_j|>M_k\}} \left[(2+c)\tilde{a}_k|\nabla u_j| + 2\tilde{b}_k\right] dx \leq \frac{1}{2k}.$$

Also, up to a bigger $\hat{j}$, we can suppose that $c\,\varepsilon_j M_k^2 \le 1/(2k)$ for every $j \ge \hat{j}$, so that (4.15) and inequalities (4.16) and (4.17) (with M_k in place of M) yield

$$\int_{B_j} \psi_k\left(\nabla u_j + \frac{\zeta(\varepsilon_j \nabla u_j)}{\varepsilon_j}\right) dx \ge \int_{B_j} \psi_k(\nabla u_j)\, dx - \frac{1}{k}. \tag{4.18}$$

Going back to (4.14), observe that

$$\int_{B_j} \delta \left|\nabla u_j + \frac{\zeta(\varepsilon_j \nabla u_j)}{\varepsilon_j}\right|^2 dx \le c\,\delta \int_\Omega 1_{B_j} |\nabla u_j|^2\, dx \le c\,\delta, \tag{4.19}$$

in view of (4.11) and the boundedness of $\{1_{B_j}\nabla u_j\}$ in $L^2(\Omega, \mathbb{R}^{d\times d})$. Inequalities (4.14), (4.18), and (4.19) give

$$\mathscr{E}_{\varepsilon_j}(u_j) \ge \int_{B_j} \psi_k(\nabla u_j)\, dx - \frac{c}{k} - c\,\delta. \tag{4.20}$$

Concentrating on the term involving ψ_k, let us write

$$\int_{B_j} \psi_k(\nabla u_j)\, dx = \int_\Omega \psi_k(\nabla u_j)\, dx - \int_{B_j^c} \psi_k(\nabla u_j)\, dx, \tag{4.21}$$

and note that

$$\int_{B_j^c} \psi_k(\nabla u_j)\, dx \le \tilde{a}_k \int_{B_j^c} |\nabla u_j|\, dx + \tilde{b}_k |B_j^c| \le \tilde{a}_k |B_j^c|^{\frac{p-1}{p}} \|\nabla u_j\|_p + \tilde{b}_k |B_j^c|,$$

where we have used Hölder's inequality. Substituting into (4.21), since $\{u_j\}$ is uniformly bounded in $W^{1,p}(\Omega, \mathbb{R}^d)$, inequality (4.20) now reads

$$\mathscr{E}_{\varepsilon_j}(u_j) \ge \int_\Omega \psi_k(\nabla u_j)\, dx - c\,\tilde{a}_k |B_j^c|^{\frac{p-1}{p}} - \tilde{b}_k |B_j^c| - \frac{c}{k} - c\,\delta. \tag{4.22}$$

We may now take the lim inf over j in (4.22) to infer that, since $|B_j^c| \to 0$ as $j \to \infty$,

$$\liminf_j \mathscr{E}_{\varepsilon_j}(u_j) \ge \int_\Omega \psi_k(\nabla u)\, dx - \frac{c}{k} - c\,\delta, \tag{4.23}$$

where we have used the fact that for each k, ψ_k is a quasiconvex function satisfying a linear growth, and therefore the functional $w \mapsto \int_\Omega \psi_k(\nabla w)\, dx$ is sequentially weakly lower semicontinuous in $W^{1,1}(\Omega, \mathbb{R}^d)$.

Having eliminated j in (4.23), we may now take the limit in k and we can deduce, by monotone convergence, that

$$\liminf_j \mathscr{E}_{\varepsilon_j}(u_j) \ge \int_\Omega f^{qc}(\nabla u)\, dx + \delta \int_\Omega |\nabla u|^2\, dx - c\,\delta.$$

However, $\nabla u \in L^2(\Omega, \mathbb{R}^{d\times d})$ by Lemma 4.2 (ii) and $u \in W_h^{1,p}$. Therefore $u \in W_h^{1,2}$ and in turn $\int_\Omega f^{qc}(\nabla u)\, dx = \overline{\mathscr{E}}(u)$. Finally, since δ is arbitrary, the Γ-lim inf inequality follows by letting $\delta \to 0$.

For the Γ-lim sup inequality, we need to establish that $\mathscr{E}''(u) \le \overline{\mathscr{E}}(u)$ for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, where we recall that

$$\mathscr{E}''(u) := \Gamma\text{-}\limsup_{j\to\infty} \mathscr{E}_{\varepsilon_j}(u) = \inf\left\{\limsup_{j\to\infty} \mathscr{E}_{\varepsilon_j}(u_j) : u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^d)\right\}.$$

In fact, it suffices to prove that $\mathscr{E}''(u) \le \overline{\mathscr{E}}(u)$ for all $u \in W_h^{1,2}$ as otherwise the result follows trivially.

First, note that the functional $\overline{\mathscr{E}}$ is continuous on $W_h^{1,2}$ in the strong topology of $W^{1,2}(\Omega, \mathbb{R}^d)$. Indeed, letting $u_k, u \in W_h^{1,2}$,

$$\begin{aligned} |\overline{\mathscr{E}}(u_k) - \overline{\mathscr{E}}(u)| &\leq \int_\Omega |f^{qc}(\nabla u_k) - f^{qc}(\nabla u)|\, dx \\ &\leq c \int_\Omega (1 + |\nabla u_k| + |\nabla u|)|\nabla u_k - \nabla u|\, dx \\ &\leq c\, \|\nabla u_k - \nabla u\|_2, \end{aligned} \tag{4.24}$$

where the second inequality follows from [10, Proposition 2.32]. This continuity property allows us to work on the smaller space $W^{1,\infty}(\Omega, \mathbb{R}^d)$.

For convenience, let us define the functional $\mathscr{F} : W^{1,\infty} \to \overline{\mathbb{R}}$ given by

$$\mathscr{F}(u) := \begin{cases} \int_\Omega f(\nabla u)\, dx, & u \in W_h^{1,\infty} \\ +\infty, & u \in W^{1,\infty}(\Omega, \mathbb{R}^d) \setminus W_h^{1,\infty}. \end{cases}$$

By Theorem 6.2, the sequential weak* lower semicontinuous envelope of $\mathscr{F}$ in $W^{1,\infty}(\Omega, \mathbb{R}^d)$ is given by

$$\overline{\mathscr{F}}(u) := \begin{cases} \int_\Omega f^{qc}(\nabla u)\, dx, & u \in W_h^{1,\infty} \\ +\infty, & u \in W^{1,\infty}(\Omega, \mathbb{R}^d) \setminus W_h^{1,\infty}. \end{cases}$$

We now claim that $\mathscr{E}''(u) \leq \overline{\mathscr{F}}(u)$ for all $u \in W^{1,\infty}(\Omega, \mathbb{R}^d)$. We first establish that

$$\mathscr{E}''(u) \leq \mathscr{F}(u), \quad \text{for all } u \in W^{1,\infty}(\Omega, \mathbb{R}^d). \tag{4.25}$$

Indeed, we may assume that $u \in W_h^{1,\infty}$ as otherwise there is nothing to prove. Note that, for $u \in W_h^{1,\infty}$, the range of ∇u is compact and, therefore, $f_{\varepsilon_j}(\nabla u(x)) \to f(\nabla u(x))$ uniformly, where f_ε, f are defined in (2.5)-(2.6). Then,

$$\begin{aligned} \mathscr{E}''(u) &:= \inf\left\{\limsup_{j\to\infty} \mathscr{E}_{\varepsilon_j}(u_j) : u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^d)\right\} \leq \limsup_{j\to\infty} \mathscr{E}_{\varepsilon_j}(u) \\ &= \lim_{j\to\infty} \frac{1}{\varepsilon_j^2} \int_\Omega W_{\varepsilon_j}(I + \varepsilon_j \nabla u(x))\, dx = \int_\Omega f(\nabla u(x))\, dx = \mathscr{F}(u), \end{aligned}$$

for every $u \in W_h^{1,\infty}$, proving (4.25). Next, note that $\mathscr{E}''$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega, \mathbb{R}^d)$, since it is an upper Γ limit, and therefore it is also sequentially weak* lower semicontinuous in $W^{1,\infty}(\Omega, \mathbb{R}^d)$. But $\overline{\mathscr{F}}$ is the largest functional below $\mathscr{F}$ enjoying this lower semicontinuity property and, hence,

$$\mathscr{E}''(u) \leq \overline{\mathscr{F}}(u), \quad \text{for all } u \in W^{1,\infty}(\Omega, \mathbb{R}^d), \tag{4.26}$$

establishing our claim. We now show that $\mathscr{E}''(u) \leq \overline{\mathscr{E}}(u)$ for all $u \in W_h^{1,2}$ proving the Γ-lim sup inequality. Let $u \in W_h^{1,2}$ and, by [4, Proposition A.2], consider a sequence $u_k \in W_h^{1,\infty}$ such that $u_k \to u$ (strongly) in $W^{1,2}(\Omega, \mathbb{R}^d)$; in particular, $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^d)$. Then, by the lower semicontinuity of $\mathscr{E}''$, and by (4.24) and (4.26),

$$\begin{aligned} \mathscr{E}''(u) &\leq \liminf_{k\to\infty} \mathscr{E}''(u_k) \leq \liminf_{k\to\infty} \overline{\mathscr{F}}(u_k) \\ &= \liminf_{k\to\infty} \int_\Omega f^{qc}(\nabla u_k(x))\, dx = \int_\Omega f^{qc}(\nabla u(x))\, dx = \overline{\mathscr{E}}(u), \end{aligned}$$

for every $u \in W_h^{1,2}$. This completes the proof. □

Proof of Theorem 2.1. Define the functionals $\mathscr{G}_\varepsilon, \overline{\mathscr{G}} : W^{1,p}(\Omega,\mathbb{R}^d) \to (-\infty,\infty]$ as

$$\mathscr{G}_\varepsilon := \mathscr{E}_\varepsilon - \mathscr{L}, \qquad \overline{\mathscr{G}} := \overline{\mathscr{E}} - \mathscr{L},$$

so that, by the definition of $\mathscr{E}_\varepsilon$ and $\overline{\mathscr{E}}$, the hypothesis of the theorem can be rewritten as

$$\lim_{\varepsilon\to 0} \mathscr{G}_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} m_\varepsilon, \qquad m_\varepsilon := \inf_{W^{1,p}_h} \mathscr{G}_\varepsilon.$$

By Theorem 2.3 and [11, Proposition 6.21], we infer that

$$\{\mathscr{G}_\varepsilon\} \quad \Gamma\text{-converges to} \quad \overline{\mathscr{G}} \quad \text{in the weak topology of} \quad W^{1,p}(\Omega,\mathbb{R}^d). \tag{4.27}$$

Moreover, from Proposition 2.2 we deduce that $\{\mathscr{G}_\varepsilon\}$ is a weakly-equicoercive sequence of functionals in $W^{1,p}(\Omega,\mathbb{R}^d)$, because $p > 1$ and, in view of Poincaré's inequality,

$$\begin{aligned}\|u\|^p_{W^{1,p}} &\le c(1 + \mathscr{G}_\varepsilon(u) + \mathscr{L}(u)) \\ &\le c(1 + \mathscr{G}_\varepsilon(u) + \|u\|_{W^{1,p}}), \quad \text{for every } u \in W^{1,p}_h.\end{aligned}$$

By standard Γ-convergence arguments, the equicoercivity of $\{\mathscr{G}\varepsilon\}$ and convergence (4.27) ensure that $m_\varepsilon \to m$ (see [11, Theorem 7.8]). Another standard argument then shows that, up to a subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega,\mathbb{R}^d)$, where u is a minimizer of $\overline{\mathscr{G}}$. □

Proof of Corollary 2.4. By using the notation (2.9)-(2.10), we can rewrite the hypothesis of the corollary as $\liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u_\varepsilon) = 0$. Therefore, from Proposition 2.2, we obtain that, up to a subsequence, $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^d)$, for some $u \in W^{1,p}(\Omega,\mathbb{R}^d)$. By Theorem 2.3, we have that indeed $u \in W^{1,2}_h$ (recall that here $\partial_D\Omega = \partial\Omega$ and $h(x) = Fx$ on $\partial\Omega$) and

$$0 \le \overline{\mathscr{E}}(u) \le \liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u_\varepsilon) = 0,$$

so that $\operatorname{sym}(\nabla u) \in \{V^{qce} = 0\}$ a.e. in Ω, since $V^{qce}(\operatorname{sym} G) = f^{qc}(G)$ for every $G \in \mathbb{R}^{d\times d}$, by Proposition 6.4. Now since $0 \le f^{qc} \le c(1 + |\cdot|^2)$, f^{qc} is $W^{1,2}$-quasiconvex by [6] (see Appendix for definitions), so that

$$0 \le V^{qce}(\operatorname{sym} F) = f^{qc}(F) \le \fint_\Omega f^{qc}(\nabla u)dx = \fint_\Omega V^{qce}(e(u))dx = 0,$$

and in turn $V^{qce}(\operatorname{sym} F) = 0$. Now, to see that $\{V^{qce} = 0\} \subseteq Q^e_2\mathcal{M}$, we note that

$$V(E) := \lim_{\varepsilon\to 0} \frac{W_\varepsilon(I + \varepsilon E)}{\varepsilon^2} \ge \lim_{\varepsilon\to 0} \frac{g_p(\operatorname{dist}(I + \varepsilon E, \mathcal{U}_\varepsilon))}{\varepsilon^2} = \lim_{\varepsilon\to 0} \frac{\operatorname{dist}^2(I + \varepsilon E, \mathcal{U}_\varepsilon)}{2\varepsilon^2}.$$

Thus, in view of Lemma 4.4 below, we have that $V(E) \ge \frac{1}{2}\operatorname{dist}^2(E,\mathcal{M})$ for every $E \in \mathbb{R}^{d\times d}_{\rm sym}$. This in particular implies that

$$V^{qce} \ge \frac{1}{2}\left(\operatorname{dist}^2(\cdot,\mathcal{M})\right)^{qce},$$

so that, by definition of $Q^e_2\mathcal{M}$, if $E \in \{V^{qce} = 0\}$ then $E \in Q^e_2\mathcal{M}$. This concludes the proof. □

Lemma 4.4. *If $E \in \mathbb{R}^{d\times d}_{\rm sym}$ and $\mathcal{U}_\varepsilon$ is given by (1.4), then*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\operatorname{dist}^2(I + \varepsilon E, \mathcal{U}_\varepsilon) = \operatorname{dist}^2(E,\mathcal{M}). \tag{4.28}$$

Proof. For a fixed $E \in \mathbb{R}^{d\times d}_{\rm sym}$, we have that

$$\begin{aligned}\operatorname{dist}(I + \varepsilon E, \mathcal{U}_\varepsilon) &\le \min_{U\in\mathcal{M}} |I + \varepsilon E - (I + \varepsilon U + o(\varepsilon))| \\ &\le \varepsilon \min_{U\in\mathcal{M}} |E - U| + o(\varepsilon).\end{aligned}$$

Hence,

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\mathrm{dist}^2(I+\varepsilon E, \mathcal{U}_\varepsilon) \leq \mathrm{dist}^2(E,\mathcal{M}). \tag{4.29}$$

Now, let $R_\varepsilon \in SO(d)$ and let $U_\varepsilon \in \mathcal{M}$ such that the distance between $I+\varepsilon E$ and $\mathcal{U}_\varepsilon$ is achieved for $R = R_\varepsilon$ and $U = U_\varepsilon$. Since $SO(d)$ and $\mathcal{M}$ are bounded sets, we have that, up to a subsequence, $R_\varepsilon \to \hat{R}$ and $U_\varepsilon \to \hat{U}$. Also,

$$\begin{aligned}\mathrm{dist}(I+\varepsilon E, \mathcal{U}_\varepsilon) &= |I+\varepsilon E - R_\varepsilon(I+\varepsilon U_\varepsilon + o(\varepsilon))| \\ &\geq |I-R_\varepsilon| - |\varepsilon E - \varepsilon R_\varepsilon U_\varepsilon + o(\varepsilon)|,\end{aligned}$$

from which we deduce that $\hat{R} = I$. Indeed, if $\hat{R} \neq I$, then by (4.29)

$$c \geq \frac{1}{\varepsilon}\mathrm{dist}(I+\varepsilon E, \mathcal{U}_\varepsilon) \geq \left\{\frac{1}{\varepsilon}|I-R_\varepsilon| - |E - R_\varepsilon U_\varepsilon + o(1)|\right\} \to \infty,$$

which is absurd. Now, since $R_\varepsilon \to I$ and $\mathbb{R}^{d\times d}_{\mathrm{skw}} := \{A \in \mathbb{R}^{d\times d} : A = -A^T\}$ is the tangent space to $SO(d)$ at I, we have that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(R_\varepsilon - I) = A, \qquad \text{for some } A \in \mathbb{R}^{d\times d}_{\mathrm{skw}}.$$

Hence,

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{dist}^2(I+\varepsilon E, \mathcal{U}_\varepsilon) &= \lim_{\varepsilon\to 0}\left|\frac{1}{\varepsilon}(I-R_\varepsilon) + E - R_\varepsilon U_\varepsilon\right|^2 \\ &= |A + E - \hat{U}|^2 \geq |A|^2 + \min_{U\in\mathcal{M}}|E-U|^2,\end{aligned} \tag{4.30}$$

where in the last passage we have also used the fact that A is orthogonal to $E - \hat{U} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. From (4.29) and (4.30), we have

$$\begin{aligned}|A|^2 + \mathrm{dist}^2(E,\mathcal{M}) &\leq \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{dist}^2(I+\varepsilon E, \mathcal{U}_\varepsilon) \\ &\leq \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{dist}^2(I+\varepsilon E, \mathcal{U}_\varepsilon) \leq \mathrm{dist}^2(E,\mathcal{M}),\end{aligned}$$

which implies $A = 0$ and, in turn, implies (4.28). □

Remark 4.1. A simple case of Lemma 4.4 is when $\mathcal{M}$ contains only the zero matrix, so that (4.28) yields

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{dist}^2(I+\varepsilon E,\, SO(d)) = |E|^2. \tag{4.31}$$

5. Young measure representation and strong convergence

In this section we wish to discuss two improvements of our results, in the spirit of [23, Corollary 2.5 and Theorem 3.1], that is a Young measure representation of the limiting functional $\overline{\mathscr{E}}$ and the strong convergence of recovery sequences.

Recall that given a sequence $\{u_j\} \subset W^{1,p}(\Omega,\mathbb{R}^d)$, the set $B_j \subseteq \Omega$ is defined as

$$B_j := \left\{x \in \Omega : |\nabla u_j(x)| \leq \frac{1}{\sqrt{\varepsilon_j}}\right\},$$

and

$$v_j(x) = x + \varepsilon_j u_j(x),$$

where $\{\varepsilon_j\}$ is a vanishing sequence. We start with a lemma, which will enter into the following discussion.

Lemma 5.1. *Let $u \in W^{1,2}_h$ and suppose that $\{u_j\}$ is a recovery sequence for u. Then*

$$\left\{ 1_{B_j} \frac{\mathrm{dist}^2(\nabla v_j, SO(d))}{\varepsilon_j^2} \right\}$$

is equiintegrable.

Proof. For notational convenience, let $\rho_j := 1_{B_j} \varepsilon_j^{-2} \mathrm{dist}^2(\nabla v_j, SO(d))$ and suppose for contradiction that $\{\rho_j\}$ is not equiintegrable, i.e. there exists some $\alpha > 0$ such that for all k there exists j_k with

$$\int_{\left\{x\in\Omega:\rho_{j_k}(x)\geq k\right\}} \rho_{j_k}\,dx \geq \alpha.$$

Fixing any $M > 0$, we have that $\int_{\left\{x\in\Omega:\rho_{j_k}(x)\geq M\right\}} \rho_{j_k}\,dx \geq \alpha$. In particular, up to passing to the subsequence ρ_{j_k}, we may assume

$$\liminf_{j\to\infty} \int_{\{x\in\Omega:\rho_j\geq M\}} \rho_j\,dx \geq \alpha, \qquad \text{for all } M > 0. \tag{5.1}$$

Since $\{u_j\}$ is a recovery sequence for $u \in W^{1,2}_h$, we have that $\overline{\mathscr{E}}(u) = \int_\Omega f^{qc}(\nabla u)\,dx = \lim_j \mathscr{E}_{\varepsilon_j}(u_j)$. However, for j large enough

$$\mathscr{E}_{\varepsilon_j}(u_j) \geq \int_{\{x\in B_j:|\nabla u_j|<M\}} \frac{W_{\varepsilon_j}(\nabla v_j)}{\varepsilon_j^2}\,dx + c\int_{\{x\in B_j:|\nabla u_j|\geq M\}} \frac{\mathrm{dist}^2(\nabla v_j,\mathcal{U}_{\varepsilon_j})}{2\varepsilon_j^2}\,dx, \tag{5.2}$$

because $W_{\varepsilon_j}(\nabla v_j) \geq c\,g_p(\mathrm{dist}(\nabla v_j,\mathcal{U}_{\varepsilon_j}))$ and $\mathrm{dist}(\nabla v_j,\mathcal{U}_{\varepsilon_j}) \leq 1$ for a.e. $x \in B_j$ and every j large enough. Next, note that for a.e. x, there exists a rotation $R_j(x)$ and a matrix $U_j(x) \in \mathcal{M}$ such that $\mathrm{dist}(\nabla v_j(x),\mathcal{U}_{\varepsilon_j}) \geq |\nabla v_j(x) - R_j(x)| - |\varepsilon_j U_j(x) + o(\varepsilon_j)|$. But then,

$$\mathrm{dist}^2(\nabla v_j(x),\mathcal{U}_{\varepsilon_j}) \geq \frac{\mathrm{dist}^2(\nabla v_j(x), SO(d))}{2} - c\,\varepsilon_j^2 \tag{5.3}$$

and combining (5.2) with (5.3) we deduce that

$$\begin{aligned}\mathscr{E}_{\varepsilon_j}(u_j) \geq& \int_{\{x\in B_j:|\nabla u_j|<M\}} \frac{W_{\varepsilon_j}(\nabla v_j)}{\varepsilon_j^2}\,dx \\ &+ c\int_{\{x\in\Omega:|\nabla u_j|\geq M\}} \frac{\rho_j}{4}\,dx - c\,|\{x\in B_j : |\nabla u_j|\geq M\}|. \end{aligned}\tag{5.4}$$

Also, note that $|\nabla u_j| = \varepsilon_j^{-1}|\nabla v_j - I| \geq \varepsilon_j^{-1}\mathrm{dist}(\nabla v_j, SO(d))$, so that

$$\left\{x \in B_j : \rho_j(x) \geq M^2\right\} \subseteq \{x\in B_j : |\nabla u_j| \geq M\},$$

and that $|\{x \in B_j : |\nabla u_j| \geq M\}| \leq c/M$. Hence, (5.4) becomes

$$\mathscr{E}_{\varepsilon_j}(u_j) \geq \int_{\{x\in B_j:|\nabla u_j|<M\}} \frac{W_{\varepsilon_j}(\nabla v_j)}{\varepsilon_j^2}\,dx + c\int_{\{x\in\Omega:\rho_j(x)\geq M^2\}} \frac{\rho_j}{4}\,dx - \frac{c}{M}. \tag{5.5}$$

Let us now consider the first term on the right side of (5.5). Proceeding as in the proof of the Γ-lim inf inequality in Theorem 2.3 (to which we refer for the notation), we infer that for some arbitrary δ,

$$\int_{\{x\in B_j:|\nabla u_j|<M\}} \frac{W_{\varepsilon_j}(\nabla v_j)}{\varepsilon_j^2}\,dx \geq \int_{\{x\in B_j:|\nabla u_j|<M\}} \psi_k(\nabla u_j)\,dx - \frac{c}{k} - c\,\delta, \tag{5.6}$$

for every $j \geq \hat{j} = \hat{j}(\delta, k)$. Also, we have that

$$\begin{aligned}\int_{\{x\in B_j : |\nabla u_j| < M\}} \psi_k(\nabla u_j)\,dx &\geq \int_{B_j} \psi_k(\nabla u_j)\,dx \\ &\quad - \int_{\{x\in B_j : |\nabla u_j| \geq M\}} (\tilde{a}_k|\nabla u_j| + \tilde{b}_k)\,dx \\ &\geq \int_{B_j} \psi_k(\nabla u_j)\,dx - \frac{c}{M}\tilde{a}_k - \frac{c}{M}\tilde{b}_k. \end{aligned} \tag{5.7}$$

By inequalities (5.6) and (5.7), we can rewrite (5.5) as

$$\begin{aligned}\mathscr{E}_{\varepsilon_j}(u_j) \geq \int_{B_j} \psi_k(\nabla u_j)\,dx - \frac{c}{M}\tilde{a}_k - \frac{c}{M}\tilde{b}_k - \frac{c}{k} - c\,\delta \\ + c \int_{\{x\in\Omega : \rho_j(x) \geq M^2\}} \frac{\rho_j}{4}\,dx - \frac{c}{M}.\end{aligned}$$

Taking the limit as $j \to \infty$ and using (5.1) gives

$$\int_\Omega f^{qc}(\nabla u)\,dx \geq \int_\Omega \psi_k(\nabla u)\,dx + c\,\alpha - \frac{c}{M}\tilde{a}_k - \frac{c}{M}\tilde{b}_k - \frac{c}{k} - c\,\delta - \frac{c}{M},$$

and letting $M \to \infty$, we obtain

$$\int_\Omega f^{qc}(\nabla u)\,dx \geq \int_\Omega \psi_k(\nabla u)\,dx + c\,\alpha - \frac{c}{k} - c\,\delta. \tag{5.8}$$

Next, taking the limit $k \to \infty$ in (5.8), by monotone convergence we get that

$$\int_\Omega f^{qc}(\nabla u)\,dx \geq \int_\Omega f^{qc}(\nabla u)\,dx + \delta \int_\Omega |\nabla u|^2 dx + c\,\alpha - c\,\delta,$$

and by the arbitrariness of δ we deduce that

$$\int_\Omega f^{qc}(\nabla u)\,dx \geq \int_\Omega f^{qc}(\nabla u)\,dx + c\,\alpha,$$

which is absurd since $c\,\alpha > 0$. This contradiction concludes the proof. □

The next result says that the Young measure representation of the limiting functional $\overline{\mathscr{E}}$ at a point $u \in W^{1,2}_h$ can be obtained through the Young measure generated by the gradients of a recovery sequence for u.

Proposition 5.2. *Under the conditions of Theorem 2.1, if $\{u_j\}$ is a recovery sequence for $u \in W^{1,2}_h$. Then we have the representation*

$$\overline{\mathscr{E}}(u) = \int_\Omega \int_{\mathbb{R}^{d\times d}} f(F) d\nu_x(F) dx,$$

where $(\nu_x)_{x\in\Omega}$ is the p-gradient Young measure generated by the sequence $\{\nabla u_j\}$.

Before proving this result, some comments are in order. When the growth condition (2.1) with $p = 2$ is assumed, proving the Young measure representation can be reduced to showing that a recovery sequence $\{u_j\}$ is indeed a relaxing sequence for the limiting functional. In turn, this amounts to the equiintegrability of $\{|\nabla u_j|^2\}$, which can be deduced from the rigidity results of [9] and the equiintegrability of $\{\mathrm{dist}^2(\nabla v_j, SO(d))/\varepsilon_j^2\}$ (see [23, Lemma 4.2]). For $1 < p < 2$, one cannot hope to prove the latter; instead, the previous lemma says that the equiintegrability of $\{1_{B_j}\mathrm{dist}^2(\nabla v_j, SO(d))/\varepsilon_j^2\}$ holds true. Nevertheless, this cannot be used to prove the equiintegrability of the sequence $\{|1_{B_j}\nabla u_j|^2\}$ (which could play the role of $\{|\nabla u_j|^2\}$) via a straightforward application of [9]. The reason is essentially that $1_{B_j}\nabla u_j$ is not itself a gradient and we cannot apply the rigidity results, at least not in any obvious way. Note, however, that recovery sequences are indeed p-equiintegrable as the following remark points out.

Remark 5.1. As done in [4], it is possible to prove that the sequence

$$\left\{\frac{\operatorname{dist}^p(\nabla v_j, SO(d))}{\varepsilon_j^p}\right\}$$

is equiintegrable. An application of [9] then gives the equiintegrability of $\{|\nabla u_j|^p\}$. However, this is not enough for the Young measure representation due to the quadratic growth of f.

In view of the previous discussion, a different strategy needs to be sought in order to prove the Young measure representation. The idea is to show that, given a recovery sequence $\{u_j\}$, one can construct a "relaxing" sequence for the limiting functional via a suitable truncation of ∇u_j. This is the content of the following lemma, where the truncation operator $T_k : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is defined as

$$T_k(F) := \begin{cases} k\frac{F}{|F|}, & |F| > k \\ \\ F, & |F| \le k. \end{cases}$$

Lemma 5.3. *Under the conditions of Theorem 2.1, if* $\{u_j\}$ *is a recovery sequence for* $u \in W_h^{1,2}$*, there exists a subsequence* $\{u_{j_k}\} \subset \{u_j\}$ *such that the truncations* $Y_k : \Omega \to \mathbb{R}^{d\times d}$ *defined as*

$$Y_k := T_k(\nabla u_{j_k})$$

satisfy

$$\int_\Omega f(Y_k)dx \longrightarrow \int_\Omega f^{qc}(\nabla u)dx. \tag{5.9}$$

Proof. Let $\{u_j\}$ be a recovery sequence for u, so that $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^d)$ and $\mathscr{E}_{\varepsilon_j}(u_j) \to \overline{\mathscr{E}}(u)$, and let $(\nu_x)_{x\in\Omega}$ be the p-gradient Young measure generated by $\{\nabla u_j\}$. We split the proof into several steps.

Step 1. We prove that $(\nu_x)_{x\in\Omega}$ is a 2-gradient Young measure. Indeed, since

$$\big\{x \in \Omega : \nabla u_j(x) \neq 1_{B_j}\nabla u_j(x)\big\} = B_j^c$$

and $|B_j^c| \to 0$, the sequences $\{\nabla u_j\}$ and $\big\{1_{B_j}\nabla u_j\big\}$ generate the same Young measure $(\nu_x)_{x\in\Omega}$ (see e.g. [22, Lemma 6.3]). In particular, by [22, Theorem 6.11], we have that

$$\int_\Omega \langle \nu_x, |\cdot|^2\rangle\, dx := \int_\Omega\int_{\mathbb{R}^{d\times d}} |A|^2 d\nu_x(A)\, dx \le \liminf_{j\to\infty}\int_\Omega |1_{B_j}\nabla u_j|^2\, dx \le c.$$

This bound, together with the fact that $(\nu_x)_{x\in\Omega}$ is a p-gradient Young measure concludes the proof of Step 1, in view of [20, Corollary 1.8].

Step 2. For every subsequence $\{u_{j_k}\} \subset \{u_j\}$, the sequence $\{Y_k\}$ defined as in the statement generates the same Young measure $(\nu_x)_{x\in\Omega}$. As in Step 1, this follows directly from

$$|\{x \in \Omega : Y_k(x) \neq \nabla u_{j_k}(x)\}| = |\{x \in \Omega : |\nabla u_{j_k}(x)| > k\}| \le \frac{1}{k}\int_\Omega |\nabla u_{j_k}|dx,$$

and the boundedness of $\{\nabla u_j\}$ in $L^p(\Omega, \mathbb{R}^{d\times d})$.

Step 3. Here we show that there exists a subsequence $\{u_{j_k}\} \subset \{u_j\}$ such that $\{|Y_k|^2\}$ is equiintegrable. Note that

$$\lim_{k\to\infty}\lim_{j\to\infty}\int_\Omega |T_k(\nabla u_j)|^2\, dx = \lim_{k\to\infty}\int_\Omega \langle \nu_x, |T_k(\cdot)|^2\rangle\, dx = \int_\Omega \langle \nu_x, |\cdot|^2\rangle\, dx,$$

where the first equality follows from the fact that, for every k, $\{|T_k(\nabla u_j)|^2\}_j$ is equiintegrable and the second equality from monotone convergence. Hence, for a diagonal subsequence $\{j_k\}$, the convergence

$$\int_\Omega |Y_k|^2 dx = \int_\Omega |T_k(\nabla u_{j_k})|^2 dx \longrightarrow \int_\Omega \langle \nu_x, |\cdot|^2\rangle dx$$

holds as $k \to \infty$. Standard arguments then imply that $\{|Y_k|^2\}$ is equiintegrable, in view of the fact that $(\nu_x)_{x\in\Omega}$ is 2-gradient Young measure.

Step 4. We now conclude the proof of the lemma. On the one hand, by [22, Theorem 6.11] and the standard characterization of gradient Young measures,

$$\liminf_{k\to\infty} \int_\Omega f(Y_k)\,dx \geq \int_\Omega \langle \nu_x, f\rangle\,dx \geq \int_\Omega \langle \nu_x, f^{qc}\rangle\,dx \geq \int_\Omega f^{qc}(\nabla u)\,dx. \tag{5.10}$$

We are left to show that

$$\limsup_{k\to\infty} \int_\Omega f(Y_k)\,dx \leq \int_\Omega f^{qc}(\nabla u)\,dx.$$

Note that, fixing $M > 0$,

$$\begin{aligned}\int_\Omega f^{qc}(\nabla u)\,dx &\geq \limsup_{j\to\infty} \int_{\{|\nabla u_j|\leq M\}} f_{\varepsilon_j}\left(\nabla u_j + \frac{\zeta(\varepsilon_j \nabla u_j)}{\varepsilon_j}\right) dx \\ &= \limsup_{j\to\infty} \int_{\{|\nabla u_j|\leq M\}} f(\nabla u_j)\,dx,\end{aligned}$$

where f_{ε_j} and ζ are defined in (2.4) and (4.10), respectively, and the equality is due to the fact that $f_{\varepsilon_j} \to f$ uniformly on compact subsets and $\zeta(\varepsilon_j\nabla u_j)/\varepsilon_j \to 0$ uniformly on $\{|\nabla u_j| \leq M\}$. In particular, the previous inequality holds for the subsequence $\{u_{j_k}\}$ introduced in Step 3, leading to

$$\int_\Omega f^{qc}(\nabla u)\,dx \geq \limsup_{k\to\infty} \int_{\{|Y_k|\leq M\}} f(Y_k)\,dx,$$

where we have also used the fact that for k large, $Y_k = \nabla u_{j_k}$ on $\{|\nabla u_{j_k}| \leq M\}$. Finally, observe that

$$\limsup_{k\to\infty} \int_{\{|Y_k|\leq M\}} f(Y_k)\,dx \geq \limsup_{k\to\infty} \int_\Omega f(Y_k)dx - \limsup_{k\to\infty} \int_{\{|Y_k|>M\}} f(Y_k)dx$$

and that the second summand on the right-hand side vanishes as $M \to \infty$ due to Step 3 and the 2-growth of f. This concludes the proof. □

Proof of Proposition 5.2. Let $\{u_j\}$ be a recovery sequence for $u \in W^{1,2}_h$ and $(\nu_x)_{x\in\Omega}$ be the gradient Young measure associated with $\{\nabla u_j\}$. Considering the subsequence $\{u_{j_k}\}$ given by Lemma 5.3, we have that

$$\overline{\mathscr{E}}(u) := \int_\Omega f^{qc}(\nabla u)dx = \lim_{k\to\infty} \int_\Omega f(Y_k)\,dx \geq \int_\Omega \langle \nu_x, f\rangle\,dx \geq \overline{\mathscr{E}}(u),$$

where the equality follows from (5.9) and the inequalities follow from (5.10). □

Remark 5.2. Note that Proposition 5.2 can equivalently be expressed in terms of the function $V : \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}$, recalling that V is obtained as the limit of $W_\varepsilon(I+\varepsilon\cdot)/\varepsilon^2$ and $f(F) := V(\operatorname{sym} F)$. To do so, one needs to consider the measure ν^e_x defined for a.e. $x \in \Omega$ as the pushforward of ν_x under the transformation $g : F \mapsto \operatorname{sym} F$, i.e.

$$\nu^e_x(\mathcal{B}) := \nu_x(g^{-1}(\mathcal{B})),$$

for all Borel subsets $\mathcal{B}$ of $\mathbb{R}^{d\times d}_{\rm sym}$. Then, by a simple change of variables (see e.g. [16, Section 39]), we infer that

$$\begin{aligned}\mathscr{E}(u) &= \int_\Omega \int_{\mathbb{R}^{d\times d}} V(g(F))\, d\nu_x(F)\, dx \\ &= \int_\Omega \int_{\mathbb{R}^{d\times d}_{\rm sym}} V(E)\, d\nu_x(g^{-1}(E))\, dx = \int_\Omega \int_{\mathbb{R}^{d\times d}_{\rm sym}} V(E)\, d\nu^e_x(E)\, dx.\end{aligned}$$

Turning attention to the second improvement regarding the strong convergence of recovery sequences, let us first recall that a function $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ is *uniformly strictly quasiconvex* if there exists a constant $\delta > 0$ such that for every $F \in \mathbb{R}^{d\times d}$

$$\int_\Omega [f(F + \nabla\varphi) - f(F)]dx \geq \delta \int_\Omega |\nabla\varphi|^2 dx, \quad \text{for every } \varphi \in W^{1,\infty}_0(\Omega, \mathbb{R}^d). \tag{5.11}$$

We have the following proposition.

Proposition 5.4. *Under the conditions of Theorem 2.1, suppose that the limiting density f is uniformly strictly quasiconvex. If $\{u_j\}$ is a recovery sequence for $u \in W^{1,2}_h$, then $u_j \to u$ strongly in $W^{1,p}(\Omega, \mathbb{R}^d)$.*

Proof. Since f is uniformly strictly quasiconvex, Proposition 5.2 says that

$$\int_\Omega f(\nabla u)\, dx = \int_\Omega \langle \nu_x, f\rangle\, dx. \tag{5.12}$$

Also, we have that $\langle \nu_x, f\rangle \geq f(\nabla u(x))$ for a.e. $x \in \Omega$, by the characterization of gradient Young measures (see [22, Theorem 8.14]). Combining this with (5.12), we infer that

$$\langle \nu_x, f\rangle = f(\nabla u(x)), \qquad \text{for a.e. } x \in \Omega. \tag{5.13}$$

Next, we claim that

$$\langle \nu_x, |\cdot|^2\rangle = |\nabla u(x)|^2, \qquad \text{for a.e. } x \in \Omega. \tag{5.14}$$

To see this, let us first note that, since f satisfies (5.11) for some $\gamma > 0$, then the function $g(F) := f(F) - \tilde\gamma|F|^2$ is quasiconvex for every $0 < \tilde\gamma \leq \gamma$. This fact, together with the growth bounds $-\tilde\gamma|F|^2 \leq g(F) \leq C(1 + |F|^2)$ (recall that f is nonnegative and that $f(F) \leq C(1+|F|^2)$ by assumption), allows us to deduce that $\langle \nu_x, g\rangle \geq g(\nabla u(x))$ for a.e. $x \in \Omega$, which is equivalent to

$$\langle \nu_x, f\rangle - f(\nabla u(x)) \geq \gamma(\langle \nu_x, |\cdot|^2\rangle - |\nabla u(x)|^2), \quad \text{for a.e. } x \in \Omega.$$

But the left side of this inequality is zero in view of (5.13), so that $\langle \nu_x, |\cdot|^2\rangle \leq |\nabla u(x)|^2$ for a.e. $x \in \Omega$. The convexity of the map $F \mapsto |F|^2$ then gives (5.14). In particular, this implies that

$$\nu_x = \delta_{\nabla u(x)}, \qquad \text{for a.e. } x \in \Omega. \tag{5.15}$$

This is because, since $(\nu_x)_{x\in\Omega}$ is a 2-gradient Young measure, there exists a sequence $\{w_j\}$ such that $\nabla w_j \rightharpoonup \nabla u$ in $L^2(\Omega, \mathbb{R}^{d\times d})$, $\{|\nabla w_j|^2\}$ is equiintegrable and $\{\nabla w_j\}$ generates the measure $(\nu_x)_{x\in\Omega}$. By Young measure representation and (5.14), we infer that

$$\lim_{j\to\infty} \int_\Omega |\nabla w_j(x)|^2 dx = \int_\Omega \langle \nu_x, |\cdot|^2\rangle dx = \int_\Omega |\nabla u(x)|^2 dx$$

and, therefore, $\nabla w_j \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^{d\times d})$. But then $\nu_x = \delta_{\nabla u(x)}$ a.e. in Ω by e.g. [22, Proposition 6.12].

We can now conclude the proof. Again by [22, Proposition 6.12], equation (5.15) implies that, up to a subsequence, $\{1_{B_j}\nabla u_j\}$ converges to ∇u pointwise a.e. in

Ω. Also, by Lemma 4.2 (i), we have that, up to a further subsequence and for a.e. $x \in \Omega$, $1_{B_j^c}(x)\nabla u_j(x) \to 0$, so that

$$\nabla u_j(x) = 1_{B_j}(x)\nabla u_j(x) + 1_{B_j^c}(x)\nabla u_j(x) \to \nabla u(x).$$

Hence, in view of Remark 5.1, an application of Vitali's convergence theorem concludes the proof of the proposition. ☐

6. Appendix

6.1. **Relaxation results.** In this section we present a version of Theorem 6.1 below, suitable for our purposes.

Theorem 6.1 (Statement III.7, [1]). *Let $1 \le q \le +\infty$ and let $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ be a continuous function satisfying*

$$\begin{aligned} &0 \le f(F) \le c(1+|F|^q), \quad \text{if } q \in [1,\infty) \\ &f \textit{ is locally bounded}, \qquad \text{if } q = \infty. \end{aligned}$$

Define

$$\mathscr{I}(u) := \int_\Omega f(\nabla u)dx, \qquad \overline{\mathscr{I}}(u) := \int_\Omega f^{qc}(\nabla u)dx,$$

where f^{qc} is the quasiconvexification of f. Then $\overline{\mathscr{I}}$ is the seq. w.l.s.c. (w.l.s.c.) envelope of $\mathscr{I}$ in $W^{1,q}(\Omega,\mathbb{R}^d)$ ($W^{1,\infty}(\Omega,\mathbb{R}^d)$) if $q<\infty$ ($q=\infty$).*

With the aim of including boundary data, we extend Theorem 6.1 in the following way:

Theorem 6.2. *Let Ω be a bounded domain with Lipschitz boundary and let*

$$\mathscr{F}(u) = \begin{cases} \int_\Omega f(\nabla u)dx, & u \in W_h^{1,q} \\ +\infty, & u \in W^{1,q}\setminus W_h^{1,q}, \end{cases}$$

for $1 \le q \le \infty$, and $W_h^{1,q}$ defined as in (2.3). Suppose that $0 \le f(F) \le c(1+|F|^q)$, for every $F \in \mathbb{R}^{d\times d}$ when $q<\infty$ and f is locally bounded if $q=\infty$. Then the seq. w.l.s.c. envelope of $\mathscr{F}$ in $W^{1,q}$ (weak for $q=\infty$) is*

$$\overline{\mathscr{F}}(u) := \begin{cases} \int_\Omega f^{qc}(\nabla u)dx, & u \in W_h^{1,q} \\ +\infty, & u \in W^{1,q}\setminus W_h^{1,q}. \end{cases}$$

Proof. First assume $q \in [1,\infty)$. By [1], we know that $\overline{\mathscr{F}}$ is seq. w.l.s.c. in $W^{1,q}$ and $\overline{\mathscr{F}} \le \mathscr{F}$. Thus, for any $u_j \rightharpoonup u$ in $W^{1,q}$,

$$\overline{\mathscr{F}}(u) \le \liminf_{j\to\infty} \overline{\mathscr{F}}(u_j) \le \liminf_{j\to\infty} \mathscr{F}(u_j).$$

It remains to prove that there exists $u_j \rightharpoonup u$ in $W^{1,q}$ such that

$$\overline{\mathscr{F}}(u) \ge \limsup_{j\to\infty} \mathscr{F}(u_j).$$

For simplicity, we prove the case where $\Omega = B_r$ is the ball of radius r, centered at the origin. By Theorem 6.1, for any $u \in W_h^{1,q}$ there exists a sequence $\{w_j\} \in W^{1,q}(\Omega,\mathbb{R}^d)$ such that

$$\overline{\mathscr{F}}(u) = \lim_{j\to\infty} \int_\Omega f(\nabla w_j)dx.$$

We need to modify the sequence $\{w_j\}$ to account for the boundary data. In order to do this, for $0 < s < s+\varepsilon < r$, let φ be a cut-off between B_s and $B_{s+\varepsilon}$, i.e. $\varphi \ge 0$, $\varphi \equiv 1$ on B_s and $\varphi \equiv 0$ on $B_{s+\varepsilon}$, and $|\nabla\varphi| \le 1/\varepsilon$. In what follows, the parameters s and ε are chosen to depend on j. Define

$$u_j := \varphi w_j + (1-\varphi)u,$$

so that $\{u_j\} \subset W^{1,q}_h$ and $u_j \rightharpoonup u$. Then,

$$\int_{B_r} f(\nabla u_j) = \int_{B_s} f(\nabla w_j) + \int_{B_r\setminus B_{s+\varepsilon}} f(\nabla u) + \int_{B_{s+\varepsilon}\setminus B_s} f(\nabla u_j)$$
$$\leq \int_{B_s} f(\nabla w_j) + \int_{B_r\setminus B_{s+\varepsilon}} f(\nabla u) + c\int_{B_{s+\varepsilon}\setminus B_s} (1+|\nabla u_j|^q)\,. \tag{6.1}$$

From the definition of u_j, $\nabla u_j = (w_j - u)\otimes\nabla\varphi + (\nabla w_j - \nabla u)\varphi$, so that

$$|\nabla u_j|^q \leq c\left(\frac{1}{\varepsilon^q}|w_j-u|^q + |\nabla w_j|^q + |\nabla u|^q\right),$$

and in turn

$$\int_{B_r} f(\nabla u_j) \leq \int_{B_r} f(\nabla w_j) + \int_{B_r\setminus B_{s+\varepsilon}} f(\nabla u) +$$
$$c\int_{B_{s+\varepsilon}\setminus B_s}(1+|\nabla w_j|^q + |\nabla u|^q) + c\int_{B_{s+\varepsilon}\setminus B_s}\frac{1}{\varepsilon^q}|w_j-u|^q. \tag{6.2}$$

Since, up to a subsequence,

$$\int_{B_r}|w_j-u|^q \leq \frac{1}{j^{2q+1}},$$

choosing $\varepsilon = \varepsilon_j = j^{-2}$, then

$$\int_{B_{s+\varepsilon_j}\setminus B_s}\frac{1}{\varepsilon_j^q}|w_j-u|^q \leq \frac{1}{j}.$$

The choice of this subsequence, as well as of ε_j, will become clear later. Regarding the second-to-last term in (6.2), note that $\exists c>0$ such that

$$\int_{B_r}(1+|\nabla w_j|^q+|\nabla u|^q) \leq c, \qquad \text{for all } j.$$

In particular,

$$\sum_{i=0}^{j-1}\int_{B_{r-\frac{i}{j^2}}\setminus B_{r-\frac{i+1}{j^2}}}(1+|\nabla w_j|^q+|\nabla u|^q)\leq c,$$

therefore, for every j there exists i_j such that

$$\int_{B_{r-\frac{i_j}{j^2}}\setminus B_{r-\frac{i_j+1}{j^2}}}(1+|\nabla w_j|^q+|\nabla u|^q)\leq \frac{c}{j}.$$

Choosing $s_j = r - \frac{i_j+1}{j^2}$, then

$$s_j + \varepsilon_j = r - \frac{i_j}{j^2} \tag{6.3}$$

and (6.2) becomes

$$\int_{B_r} f(\nabla u_j) \leq \int_{B_r} f(\nabla w_j) + \int_{B_r\setminus B_{s_j+\varepsilon_j}} f(\nabla u) + \frac{c}{j}.$$

Since $0\leq i_j\leq j-1$ for each j, then $i_j/j^2\to 0$ and in turn from (6.3)

$$\left|B_r\setminus B_{s_j+\varepsilon_j}\right|\to 0, \qquad \text{as } j\to\infty.$$

Hence,

$$\limsup_{j\to\infty}\mathscr{F}(u_j) = \limsup_{j\to\infty}\int_{B_r} f(\nabla u_j) \leq \lim_{j\to\infty}\int_{B_r} f(\nabla w_j) = \overline{\mathscr{F}}(u).$$

This completes the proof for $q<\infty$.

When $q=\infty$ one can argue in the same way but we do not have the inequality in (6.1). However, $\{u_j\}$ is bounded in $W^{1,\infty}_h$ and the sequence $\{f(\nabla u_j)\}$ is bounded in $L^\infty(\Omega)$. Then, with the same choices for ε_j and s_j, we obtain that

$$\int_{B_{s_j+\varepsilon_j}\setminus B_{s_j}} f(\nabla u_j) \le c\left|B_{s_j+\varepsilon_j}\setminus B_{s_j}\right| \to 0.$$

□

6.2. **Quasiconvexity on linear strains.** We recall that, for $1\le q\le\infty$, a locally bounded and Borel measurable function $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ is $W^{1,q}$-*quasiconvex* if

$$f(F)\le \fint_U f(F+\nabla\varphi)dx, \qquad \text{for all } \varphi\in W^{1,q}_0(U;\mathbb{R}^m),\ F\in\mathbb{R}^{m\times n},$$

where $U\subset\mathbb{R}^n$ is bounded and open, and $\fint_U := \frac{1}{|U|}\int_U$. This definition is independent of the choice of U. As it is common in the literature, we refer to $W^{1,\infty}$-quasiconvexity as *quasiconvexity*. The *quasiconvexification* of a function $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ is defined as

$$f^{qc} := \sup\{g:\mathbb{R}^{m\times n}\to\mathbb{R} \,:\, g\le f,\ g \text{ is quasiconvex}\}, \tag{6.4}$$

with the convention that $f^{qc}\equiv-\infty$ if the above set is empty. Recall that for every locally bounded and Borel measurable $f:\mathbb{R}^{m\times n}\to\mathbb{R}$,

$$f^{qc}(\xi) = \inf_{\varphi\in W^{1,\infty}_0(U,\mathbb{R}^m)} \fint_U f(\xi+\nabla\varphi)\,dx, \tag{6.5}$$

whenever $U\subset\mathbb{R}^n$ is a bounded and open set with $|\partial U|=0$. For a proof of this characterization, in the case where the set in (6.4) is nonempty, see [10, Theorem 6.9]). Otherwise, we refer the reader to [21, Theorem 4.5].

A function $f:\mathbb{R}^{d\times d}_{\text{sym}}\to\mathbb{R}$, is *quasiconvex on linear strains* if for every $E\in\mathbb{R}^{d\times d}_{\text{sym}}$

$$f(E)\le \fint_U f(E+e(\varphi))dx, \qquad \text{for every } \varphi\in W^{1,\infty}_0(U,\mathbb{R}^d),$$

where $U\subset\mathbb{R}^d$ is open and bounded, and $e(\varphi):=\operatorname{sym}(\nabla\varphi)$. We recall that, as for quasiconvexity, this definition does not depend on U. For $f:\mathbb{R}^{d\times d}_{\text{sym}}\to\mathbb{R}$, the *quasiconvexification on linear strains* is defined as

$$f^{qce} := \sup\{g:\mathbb{R}^{d\times d}_{\text{sym}}\to\mathbb{R} \,:\, g\le f,\ g \text{ is quasiconvex on linear strains}\}.$$

In what follows we collect some known results concerning the notion of quasiconvexity on linear strains.

Lemma 6.3. *Let $V:\mathbb{R}^{d\times d}_{\text{sym}}\to\mathbb{R}$ be locally bounded and Borel measurable, and define $f:\mathbb{R}^{d\times d}\to\mathbb{R}$ by $f(F):=V(\operatorname{sym}F)$. Then f is quasiconvex if and only if V is quasiconvex on linear strains.*

Proof. The proof follows immediately from the definitions. □

Proposition 6.4. *Under the hypotheses of Lemma 6.3, $f^{qc}(F)=V^{qce}(\operatorname{sym}F)$, for every $F\in\mathbb{R}^{d\times d}$.*

Proof. By Lemma 6.3, $F\mapsto V^{qce}(\operatorname{sym}F)$ is quasiconvex. Also, $V^{qce}(\operatorname{sym}F)\le V(\operatorname{sym}F)=f(F)$, so that by definition $V^{qce}(\operatorname{sym}F)\le f^{qc}(F)$ for every $F\in\mathbb{R}^{d\times d}$. Now, fix $U\subset\mathbb{R}^d$ open and bounded and define

$$h(E) := \inf_{\varphi\in W^{1,\infty}_0(U,\mathbb{R}^d)} \fint_U V(E+e(\varphi))dx, \qquad E\in\mathbb{R}^{d\times d}_{\text{sym}}. \tag{6.6}$$

Then by (6.5)

$$h(\operatorname{sym} F) = \inf_{\varphi \in W_0^{1,\infty}(U,\mathbb{R}^d)} \fint_U f(F + \nabla\varphi)dx = f^{qc}(F).$$

Therefore, by Lemma 6.3, h is quasiconvex on linear strains. Taking $\varphi \equiv 0$ in (6.6), $h \leq V$ and then $h \leq V^{qce}$. Hence,

$$f^{qc}(F) = h(\operatorname{sym} F) \leq V^{qce}(\operatorname{sym} F), \qquad \text{for every } F \in \mathbb{R}^{d\times d}.$$

This conclues the proof. □

As an immediate consequence of Proposition 6.4, we get the following corollary.

Corollary 6.5. *Let $V : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ be locally bounded and Borel measurable. Then for all $E \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$*

$$V^{qce}(E) = \inf_{\varphi \in W_0^{1,\infty}(U,\mathbb{R}^d)} \fint_U V(E + e(\varphi))dx,$$

where $U \subset \mathbb{R}^d$ is open and bounded.

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