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GENERALIZED TWISTS AS ELASTIC ENERGY EXTREMALS ON ANNULI, QUATERNIONS AND LIFTING TWIST LOOPS TO THE SPINOR GROUPS

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Let $\mathbf{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$ be a *generalized* annulus and consider the Dirichlet energy functional

$$\mathbb{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} |\nabla u(x)|^2 dx,$$

over the space of admissible maps

$$\mathcal{A}_\varphi(\mathbf{X}) = \{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u = 1 \text{ a.e. in } \mathbf{X}, u|_{\partial \mathbf{X}} = \varphi\},$$

where φ is the *identity* map. In this paper we consider a class of maps referred to as *generalized* twists and examine them in connection with the Euler–Lagrange equation associated with $\mathbb{F}[\cdot, \mathbf{X}]$ on $\mathcal{A}_\varphi(\mathbf{X})$. The approach is novel and is based on *lifting* twist loops from $\mathbf{SO}(n)$ to its double cover $\mathbf{Spin}(n)$ and reformulating the equations accordingly. We restrict our attention to *low* dimensions and prove that for $n = 4$ the system admits *infinitely* many smooth solutions in the form of twists while for $n = 3$ this number sharply reduces to *one*. We discuss some qualitative features of these solutions in view of their remarkable *explicit* representation through the exponential map of $\mathbf{Spin}(n)$.

Keywords: Dirichlet energy; generalized twists; elastic energy extremals; scaled geodesics; $\mathbf{Spin}(n)$; $\mathbf{SO}(n)$.

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1. Introduction

Let $\mathbf{X} \subset \mathbb{R}^n$ be a *generalized* annulus and consider the energy functional

$$\mathbb{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} |\nabla u(x)|^2 dx, \tag{1.1}$$

over the space of admissible maps

$$\mathcal{A}(\mathbf{X}) = \{u \in W^{1,2}_\varphi(\mathbf{X}, \mathbb{R}^n) : \det \nabla u = 1 \text{ a.e. in } \mathbf{X}\}, \tag{1.2}$$

where

$$W^{1,2}_{\varphi}(\mathbf{X}, \mathbb{R}^n) = \{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : u|_{\partial \mathbf{X}} = \varphi\}$$

and φ denotes the *identity* map.

In [10] we initiated the task of *extremizing* the energy functional (1.1) over the space (1.2) by introducing a class of maps referred to as *generalized* twists and examining them as possible solutions to the associated system of Euler–Lagrange equations that here take the form

$$\begin{cases} \operatorname{div} \mathbf{S}[x, \nabla u(x)] = 0 & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u = \varphi & \text{on } \partial \mathbf{X}. \end{cases} \tag{1.3}$$

Note that the *divergence* operator acts *row*-wise and the tensor field $\mathbf{S}$ is defined through

$$\mathbf{S}[x, \mathbf{F}] = \mathbf{F} - \mathfrak{p}(x)\mathbf{F}^{-t} = [\mathbf{F}\mathbf{F}^t - \mathfrak{p}(x)\mathbf{I}_n]\mathbf{F}^{-t},$$

for all $x \in \mathbf{X}$ with $\mathbf{F} \in \mathbb{R}^{n \times n}$ satisfying $\det \mathbf{F} = 1$ while $\mathfrak{p}$ is a suitable Lagrange multiplier.[a]

A *generalized* twist by definition is a continuous self-map of an annulus, here taken, $\mathbf{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$, onto itself in the form

$$u : (r, \theta) \mapsto (r, \mathbf{Q}[r]\theta),$$

where $r = |x|$, $\theta = x/|x|$ and $\mathbf{Q} \in \mathbf{C}([a, b], \mathbf{SO}(n))$. The main motivation for considering such maps comes partly from geometric topology where *Dehn* twists are crucial instruments in describing the mapping class groups of *surfaces* (see [3]) and partly nonlinear elasticity where *generalized* twists have played a curious role in the multiple solution problems for *equilibrium* states and local minimizers (see [10–12]).

Subject to further *differentiability* assumption on the twist *path* $\mathbf{Q}$ and end-point conditions $\mathbf{Q}(a) = \mathbf{I}_n$ and $\mathbf{Q}(b) = \mathbf{I}_n$ (thus making it a *loop* in the pointed space $[\mathbf{SO}(n), \mathbf{I}_n]$) it can be shown that $u \in \mathcal{A}_{\varphi}(\mathbf{X})$ and that

$$\begin{aligned} \mathbb{F}[\mathbf{Q}[r]x; \mathbf{X}] &= \frac{1}{2}\int_{\mathbf{X}} |\nabla u|^2 dx \\ &= \frac{1}{2}\int_a^b \int_{\mathbb{S}^{n-1}} [n + r^2|\dot{\mathbf{Q}}\theta|^2] r^{n-1} d\mathcal{H}^{n-1}(\theta) dr \\ &= \frac{1}{2}\omega_n \left[n(b^n - a^n) + \int_a^b |\dot{\mathbf{Q}}|^2 r^{n+1} dr \right]. \end{aligned} \tag{1.4}$$

Interestingly the integral (1.4) has no θ-dependence and indeed its principal part upon a change of variables presents the Dirichlet energy of the twist loop $\mathbf{Q}$ and

[a] We refer the interested reader to [2, 10] for background, motivation and a preliminary discussion on the problem considered here.

hence amending itself to an investigation of scaled *geodesics* on the compact Lie group $\mathbf{SO}(n)$.

Whilst formulating conditions on the latter *twist* loops to subsequently grant solutions to the equilibrium equations (1.3) is of interest in this paper we follow an alternative route by reformulating the problem on the *universal* (double) cover of $\mathbf{SO}(n)$ the spinor group $\mathbf{Spin}(n)$. Investigating the resulting equations over $\mathbf{Spin}(n)$ apart from being interesting in its own right has the advantage of making the calculation of spin degrees (see [12]) of stationary loops and *closed* paths a straightforward task.

The study of the energy (1.4) and its reformulation on $\mathbf{Spin}(n)$ requires, in particular, working in the *Clifford* algebra $\mathcal{C}_n$ and exploiting its *multiplicative* structure (see [1]). For the purpose of this paper, however, we specialize, in low dimensions, specifically $n < 5$, where additionally by taking advantage of the so-called *accidental* isomorphisms one can connect $\mathbf{Spin}(n)$ to well known classical matrix groups (see [1, 4, 6]). Indeed, here $\mathbf{Spin}(3) \cong \mathbf{SU}(2) \cong \mathbf{Sp}(1) \cong \mathbb{S}^3$ and $\mathbf{Spin}(4) \cong \mathbf{SU}(2) \times \mathbf{SU}(2) \cong \mathbf{Sp}(1) \times \mathbf{Sp}(1)$ (see Appendix for $n = 2$).[b]

To briefly outline the plan of the paper we begin with some basic observations on the energy of *generalized* twists and the *lifting* problem for their twist loops. We have moved some of the lengthy discussions associated with the covering map $\gamma : \mathbf{Spin}(n) \to \mathbf{SO}(n)$ that play a crucial role in the lifting of the energy (1.4) to $\mathbf{Spin}(n)$ to the Appendix. The *infinite* family of *extremals* of the latter energy are then obtained in Secs. 3–4 through integration of the corresponding Euler–Lagrange equations. A detailed analysis of the equilibrium equations presented in Secs. 5.1 and 5.2 then reveals that among such extremals for $n = 3$ there is only a *single* one granting solution to the system (1.3) whereas in *sharp* contrast for $n = 4$ there is an *infinite* family of such extremals granting solutions to (1.3) whose structure and explicit form are described using the exponential map of $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$.

2. Generalized Twists as Self-Maps of X

Let $\mathbf{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$. A continuous self-map of $\overline{\mathbf{X}}$ is referred to as a *generalized* twist if it can be represented in *spherical* coordinates in the form

$$u : (r, \theta) \mapsto (r, \mathbf{Q}[r]\theta), \tag{2.1}$$

where $x \in \overline{\mathbf{X}}$ and $r = |x|$, $\theta = x/|x|$ and where $\mathbf{Q} \in \mathbf{C}([a, b], \mathbf{SO}(n))$. It can be verified that subject to the additional conditions

[**1**] $\mathbf{Q}[a] = \mathbf{I}_n$,
[**2**] $\mathbf{Q}[b] = \mathbf{I}_n$,
[**3**] $\mathbf{Q} \in W^{1,2}([a, b], \mathbf{SO}(n))$,

[b]Our notation for matrix groups and related Lie groups is standard and in agreement with e.g., [4, 6]. For an overview of related problems and techniques in geometric analysis, specifically, the theory of harmonic maps see [7, 9, 13].

the resulting twist u lies in the space of admissible maps $\mathcal{A}_\varphi(\mathbf{X})$. Furthermore, in this case the Dirichlet energy can be expressed as (see [10] for details)

$$\begin{aligned}\mathbb{F}[u;\mathbf{X}] &= \frac{1}{2}\int_{\mathbf{X}} |\nabla u|^2 dx \\ &= \frac{1}{2}\omega_n\left[n(b^n - a^n) + \int_a^b |\dot{\mathbf{Q}}|^2 r^{n+1} dr\right]. \end{aligned} \tag{2.2}$$

In the remainder of the paper we specialise in the cases $n = 3$ and $n = 4$. Our first goal here is to express the energy (2.2) and its associated Euler–Lagrange equations over the *spinor* groups $\mathbf{Spin}(3)$ and $\mathbf{Spin}(4)$ serving as the universal covering spaces of $\mathbf{SO}(3)$ and $\mathbf{SO}(4)$, respectively.[c]

3. Spin(3) ≅ Sp(1) and the Dirichlet Energy

Let $q \in \mathbf{C}([a,b], \mathbf{Sp}(1))$ and consider the associated generalized twist

$$u(x) = \mathbf{Q}_q[r]x, \tag{3.1}$$

with $r = |x|$ where $\mathbf{Q}_q$ is as described in (A.1). Then, it follows from [**1**]–[**3**] above and Lemma A.1 that $u \in \mathcal{A}(\mathbf{X})$ provided that the following hold:

[**1**] $q(a) = 1$,
[**2**] $q(b) \in \{\pm 1\}$,
[**3**] $q \in W^{1,2}([a,b], \mathbf{Sp}(1))$.

Furthermore, for the twist u defined above using (2.2) and Lemma A.1 we have that

$$\mathbb{F}[u;\mathbf{X}] = \frac{2\pi}{3}\left[3(b^3 - a^3) + 8\int_a^b |\dot{q}|^2 r^4 dr\right]. \tag{3.2}$$

Now, motivated by the above representation in what follows we consider the principal part of the energy, i.e.

$$\phi[q] := \int_a^b |\dot{q}|^2 r^4 \, dr,$$

over the space of admissible paths

$$\pi = \pi[a,b] := \left\{ q : \begin{aligned} & q(a) = 1, \\ & q(b) \in \{\pm 1\}, \\ & q \in W^{1,2}([a,b], \mathbf{Sp}(1)) \end{aligned} \right\}.$$

[c] At this stage the reader is encouraged to consult the Appendix where some of the notation and basic results used in the subsequent sections are introduced and proved.

The initial aim is to derive the Euler–Lagrange equation associated with the energy ϕ over the space $\pi[a,b]$. We claim that this takes the form

$$\frac{d}{dr}\left(r^4 \frac{dq}{dr}\bar{q}\right) = 0, \tag{3.3}$$

on $]a,b[$. Indeed fix q and for $\varepsilon \in \mathbb{R}$ put $q_\varepsilon = q + \varepsilon\xi q$ where $\xi \in \mathbf{C}_0^\infty(]a,b[,\mathbb{H}_0)$ is arbitrary [note that $\mathbb{H}_0$ denotes the set of *pure* quaternions (see Appendix A)]. Then, it can be easily seen that

$$\begin{aligned} q_\varepsilon \bar{q}_\varepsilon &= (q+\varepsilon\xi q)\overline{(q+\varepsilon\xi q)} \\ &= q\bar{q}[1+\varepsilon(\xi+\bar{\xi})+\varepsilon^2\xi\bar{\xi}] = 1+\varepsilon^2|\xi|^2, \end{aligned}$$

indicating that up to the first order q_ε is $\mathbf{Sp}(1)$-valued. Therefore (with a slight abuse of notation), we have that

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\phi[q_\varepsilon]\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\left[\int_a^b |\dot{q}_\varepsilon|^2 r^4 dr\right]\bigg|_{\varepsilon=0} \\ &= \int_a^b [(\dot{\xi}q+\xi\dot{q})\dot{\bar{q}} + \dot{q}\overline{(\dot{\xi}q+\xi\dot{q})}]r^4 dr \\ &= \int_a^b [(\dot{\xi}q\dot{\bar{q}} + \dot{q}\bar{q}\dot{\bar{\xi}}) + (\xi\dot{q}\dot{\bar{q}} + \dot{q}\dot{\bar{q}}\bar{\xi})]r^4 dr =: \mathbf{E}_1 + \mathbf{E}_2. \end{aligned}$$

We now proceed by evaluating each term separately. Indeed, with regards to the first term using the fact that $q\dot{\bar{q}}$ and $\dot{\xi}$ are pure quaternionic-valued we can write

$$\begin{aligned} \mathbf{E}_1 &:= \int_a^b (\dot{\xi}q\dot{\bar{q}} + \dot{q}\bar{q}\dot{\bar{\xi}})r^4 dr \\ &= \int_a^b -(\dot{\xi}\dot{q}\bar{q} + \dot{q}\bar{q}\dot{\xi})r^4 dr \\ &= \int_a^b \left[\xi\frac{d}{dr}(r^4\dot{q}\bar{q}) + \frac{d}{dr}(r^4\dot{q}\bar{q})\xi\right] dr. \end{aligned}$$

Note that in concluding the last line we have used integration by parts together with the boundary conditions $\xi(a) = \xi(b) = 0$. On the other hand for the second term a direct verification reveals that

$$\begin{aligned} \mathbf{E}_2 &:= \int_a^b (\xi\dot{q}\dot{\bar{q}} + \dot{q}\dot{\bar{q}}\bar{\xi})r^4 dr \\ &= \int_a^b |\dot{q}|^2(\xi+\bar{\xi})r^4 dr = 0, \end{aligned}$$

where in concluding the second equality we have used the fact that $\dot{q}\dot{\bar{q}}$ is real-valued and ξ is *pure* quaternionic-valued. Therefore, *summarizing*, by combining the latter we have that

$$
\begin{aligned}
0 &= \frac{d}{d\varepsilon}\phi[q_\varepsilon]\bigg|_{\varepsilon=0} \\
&= \int_a^b \left[\xi \frac{d}{dr}(r^4 \dot{q}\bar{q}) + \frac{d}{dr}(r^4 \dot{q}\bar{q})\xi\right] dr \\
&= \int_a^b 2\Re \left[\frac{d}{dr}(r^4 \dot{q}\bar{q})\xi\right] dr,
\end{aligned}
$$

which justifies the asserted claim.

Theorem 3.1. ($n = 3$) *The solution to the Euler–Lagrange equation associated with the energy ϕ over the space $\pi[a, b]$ is given by the quaternionic exponential*

$$
q(r) = \mathbf{Exp}\left(\frac{1}{3}\left[\frac{1}{a^3} - \frac{1}{r^3}\right]\alpha\right), \tag{3.4}
$$

where $\alpha \in \mathbb{H}_0$ is constant and has the form

$$
\alpha = \rho_\alpha \omega_\alpha,
$$

where

$$
\rho_\alpha = 3\pi \frac{a^3 b^3}{(b^3 - a^3)} m, \tag{3.5}
$$

with $m \in \mathbb{Z}$ while $\omega_\alpha \in \mathbb{H}_0$ and $|\omega_\alpha| = 1$ (i.e. ω_α is a unit pure quaternion).

Proof. As q is a solution to (3.3) there exits $\alpha \in \mathbb{H}_0$ constant such that $r^4 \dot{q}\bar{q} = \alpha$ [notice that $\dot{q}\bar{q}$ is pure quaternionic-valued] and therefore

$$
\frac{dq}{dr} = \frac{1}{r^4}\alpha q.
$$

Integrating the above equation gives

$$
q(r) = \mathbf{Exp}(-3^{-1} r^{-3}\alpha) q_\circ,
$$

where $q_\circ \in \mathbf{Sp}(1)$ is constant and *fixed*. Referring to the end-point conditions it is evident that

$$
\begin{aligned}
q(a) = 1 &\Leftrightarrow \mathbf{Exp}(-3^{-1} a^{-3}\alpha) q_\circ = 1 \\
&\Leftrightarrow q_\circ = \mathbf{Exp}(3^{-1} a^{-3}\alpha)
\end{aligned}
$$

and similarly upon substitution

$$
\begin{aligned}
q(b) = \pm 1 &\Leftrightarrow \mathbf{Exp}(-3^{-1} b^{-3}\alpha) q_\circ = \pm 1 \\
&\Leftrightarrow \mathbf{Exp}(s\alpha) = \pm 1,
\end{aligned}
$$

where for the sake of brevity and convenience we have set

$$
s := \frac{1}{3}\frac{(b^3 - a^3)}{a^3 b^3}.
$$

As $\alpha = 0$ corresponds to $q \equiv 1$ in what follows we restrict our attention to *non-zero* α. Hence to conclude it suffices to note that[d]

$$\begin{aligned}\mathbf{Exp}(s\rho_\alpha\omega_\alpha) = \pm 1 &\Leftrightarrow \cos(s\rho_\alpha) + \omega_\alpha \sin(s\rho_\alpha) = \pm 1\\ &\Leftrightarrow \cos(s\rho_\alpha) + \omega_\alpha \sin(s\rho_\alpha) = \pm 1\\ &\Leftrightarrow \rho_\alpha \in \frac{\pi}{s}\mathbb{Z},\end{aligned}$$

which is the required conclusion. □

The immediate aim next is to find the $\mathbf{SO}(3)$-valued map $\mathbf{Q}_q$ associated with the solution q from the above theorem [in the spirit of that described by (A.1)]. To this end it is convenient to set

$$g(r) := \frac{2}{3}\frac{(r^3 - a^3)}{a^3 r^3}\rho_\alpha,$$

with $r \in [a, b]$. Then, it is evident that for $\alpha = \rho_\alpha\omega_\alpha$ with ω_α being a unit pure quaternion we have that

$$\begin{aligned}q(r) &= \mathbf{Exp}\left(\frac{1}{3}\frac{(r^3 - a^3)}{a^3 r^3}\rho_\alpha\omega_\alpha\right)\\ &= \mathbf{Exp}\left(\frac{1}{2}g(r)\omega_\alpha\right)\\ &= \cos\frac{g(r)}{2} + \omega_\alpha \sin\frac{g(r)}{2}.\end{aligned}$$

Now, again utilizing the fact that ω_α is a unit pure quaternion there exists $p \in \mathbf{Sp}(1)$ such that

$$\overline{\omega_\alpha} = pk\bar{p}.$$

As a result we can write

$$\begin{aligned}q(r) &= \cos\frac{g(r)}{2} + \omega_\alpha \sin\frac{g(r)}{2}\\ &= p\left(\cos\frac{g(r)}{2} - k\sin\frac{g(r)}{2}\right)\bar{p}\\ &= p\mathbf{Exp}\left(-\frac{1}{2}g(r)k\right)\bar{p}\\ &=: p\eta(r)\bar{p}.\end{aligned}$$

Therefore, the $\mathbf{SO}(3)$-valued map $\mathbf{Q}_q$ associated with the solution q given by (3.4) in Theorem 3.1 can be written in the *diagonalized* form

$$\mathbf{Q}_q = \mathbf{P}\mathbf{Q}_\eta\mathbf{P}^t, \tag{3.7}$$

[d]It is well-known that for any real ρ (i.e. $\rho \in \mathbb{R}$) and any unit pure quaternion ω (i.e. $\omega \in \mathbb{H}_0$ with $|\omega| = 1$) the quaternionic exponential satisfies the following version of Euler's identity:

$$\mathbf{Exp}(\rho\omega) = \cos\rho + \omega\sin\rho. \tag{3.6}$$

where $\mathbf{P} = \mathbf{Q}_p \in \mathbf{SO}(3)$ is *constant* and the $\mathbf{SO}(3)$-valued map $\mathbf{Q}_\eta$ [with $\eta(r) = \mathbf{Exp}(-\frac{1}{2}g(r)k)$] has the specific form

$$[\mathbf{Q}_\eta] =: \mathrm{diag}(\mathbf{R}[g], 1)$$
$$= \begin{bmatrix} \cos g & \sin g & 0 \\ -\sin g & \cos g & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.8}$$

corresponding to a planar twist.

Proposition 3.1. *Let $q = q(r; a, b, \alpha)$ denote the solution to the Euler–Lagrange equation associated with the energy ϕ over the space $\pi[a, b]$. (See Theorem 3.1.) Then, with the notation used in the theorem we have that*

$$\phi[q] = 3\pi^2 \frac{a^3 b^3}{(b^3 - a^3)} m^2.$$

Proof. In view of Theorem 3.1 a straightforward calculation with $q = q(r)$ as in (3.4) gives

$$\phi[q] = \int_a^b |\dot{q}|^2 r^4 dr = \int_a^b \frac{1}{r^4} |\alpha q|^2 dr$$
$$= \frac{1}{3} \frac{(b^3 - a^3)}{a^3 b^3} |\alpha|^2.$$

Therefore, in view of $\alpha = \alpha(m) = \rho_\alpha \omega_\alpha$ we can write

$$\phi[q] = \frac{1}{3} \frac{(b^3 - a^3)}{a^3 b^3} |\rho_\alpha|^2 = 3\pi^2 \frac{a^3 b^3}{(b^3 - a^3)} m^2$$

which is the required conclusion. □

Now, for the twist associated with the solution q from Theorem 3.1 we have the corresponding Dirichlet energy given by

$$\mathbb{F}[u; \mathbf{X}] = 2\pi \left[(b^3 - a^3) + 8\pi^2 \frac{a^3 b^3}{(b^3 - a^3)} m^2 \right]. \tag{3.9}$$

4. Spin(4) ≅ Sp(1) × Sp(1) and the Dirichlet Energy

Let $(p, q) \in \mathbf{C}([a, b], \mathbf{Sp}(1) \times \mathbf{Sp}(1))$ and consider the associated generalized twist

$$u(x) = \mathbf{Q}_{p,q}[r]x, \tag{4.1}$$

with $r = |x|$ where $\mathbf{Q}_{p,q}$ is as described in (A.3). Then, similar to the case $n = 3$ by utilizing Lemma A.2 it follows that $u \in \mathcal{A}(\mathbf{X})$ provided that the following hold:

[**1**] $(p(a), q(a)) = (1, 1)$,
[**2**] $(p(b), q(b)) \in \{\pm(1, 1)\}$,
[**3**] $(p, q) \in W^{1,2}([a, b], \mathbf{Sp}(1) \times \mathbf{Sp}(1))$.

Furthermore, for the twist u defined above using (2.2) and Lemma A.2 we have that

$$\mathbb{F}[u, \mathbf{X}] = \pi^2 \left[(b^4 - a^4) + \int_a^b (|\dot{p}|^2 + |\dot{q}|^2) r^5 dr \right]. \tag{4.2}$$

Now, motivated by the above representation in what follows we consider the principal part of the energy, i.e.

$$\phi[(p,q)] := \int_a^b (|\dot{p}|^2 + |\dot{q}|^2) r^5 dr,$$

over the space of admissible paths

$$\pi = \pi[a,b] := \left\{ \begin{aligned} &(p(a), q(a)) = (1,1), \\ (p,q) : &(p(b), q(b)) \in \{\pm(1,1)\}, \\ &(p,q) \in W^{1,2}([a,b], \mathbf{Sp}(1) \times \mathbf{Sp}(1)) \end{aligned} \right\}.$$

Again, we proceed by deriving the Euler–Lagrange equation associated with the energy ϕ over the space $\pi[a,b]$. We claim that this takes the form

$$\frac{d}{dr}\left[r^5 \left(\frac{dp}{dr}\bar{p}, \frac{dq}{dr}\bar{q} \right) \right] = (0,0), \tag{4.3}$$

on $]a,b[$. Indeed, fix p and for $\varepsilon \in \mathbb{R}$ put $p_\varepsilon = p + \varepsilon\xi p$, where $\xi \in \mathbf{C}_0^\infty(]a,b[, \mathbb{H}_0)$ is arbitrary. Then, it can be easily seen that

$$\begin{aligned} p_\varepsilon \bar{p}_\varepsilon &= (p + \varepsilon\xi p)\overline{(p + \varepsilon\xi p)} \\ &= p\bar{p}[1 + \varepsilon(\xi + \bar{\xi}) + \varepsilon^2 \xi\bar{\xi}] = 1 + \varepsilon^2 |\xi|^2 \end{aligned}$$

indicating that up to the first order (p_ε, q) is $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$-valued. Thus (with a slight abuse of notation), we have that

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\phi[(p_\varepsilon, q)]\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\left[\int_a^b (|\dot{p}_\varepsilon|^2 + |\dot{q}|^2) r^5 dr \right]\bigg|_{\varepsilon=0} \\ &= \int_a^b [(\dot{\xi}p + \xi\dot{p})\dot{\bar{p}} + \dot{p}\overline{(\dot{\xi}p + \xi\dot{p})}] r^5 dr \\ &= \int_a^b [(\dot{\xi}p\dot{\bar{p}} + \dot{p}\bar{p}\dot{\bar{\xi}}) + (\xi\dot{p}\dot{\bar{p}} + \dot{p}\dot{\bar{p}}\bar{\xi})] r^5 dr =: \mathbf{E}_1 + \mathbf{E}_2. \end{aligned}$$

We now proceed by evaluating each term separately. Indeed, with regards to the first term using the fact that $p\dot{\bar{p}}$ and $\dot{\xi}$ are pure quaternionic-valued we can write

$$\begin{aligned}\mathbf{E}_1 &:= \int_a^b (\dot{\xi} p \dot{\bar{p}} + \dot{p}\bar{p}\dot{\bar{\xi}}) r^5 \, dr \\ &= \int_a^b -(\dot{\xi}\dot{p}\bar{p} + \dot{p}\bar{p}\dot{\xi}) r^5 \, dr \\ &= \int_a^b \left[\xi \frac{d}{dr}(r^5 \dot{p}\bar{p}) + \frac{d}{dr}(r^5 \dot{p}\bar{p})\xi \right] dr.\end{aligned}$$

Note that in concluding the last line we have used integration by parts together with the boundary conditions $\xi(a) = \xi(b) = 0$. On the other hand, for the second term a direct verification reveals that

$$\begin{aligned}\mathbf{E}_2 &:= \int_a^b (\xi \dot{p} \dot{\bar{p}} + \dot{p}\dot{\bar{p}}\bar{\xi}) r^5 \, dr \\ &= \int_a^b |\dot{p}|^2 (\xi + \bar{\xi}) r^5 \, dr = 0,\end{aligned}$$

where in concluding the second equality we have used the fact that $\dot{p}\dot{\bar{p}}$ is real-valued and ξ is *pure* quaternionic-valued. Therefore, *summarizing*, by combining the latter we have that

$$\begin{aligned}0 &= \frac{d}{d\varepsilon}\phi[(p_\varepsilon, q)]\bigg|_{\varepsilon=0} \\ &= \int_a^b \left[\xi \frac{d}{dr}(r^5 \dot{p}\bar{p}) + \frac{d}{dr}(r^5 \dot{p}\bar{p})\xi \right] dr \\ &= \int_a^b 2\Re \left[\frac{d}{dr}(r^5 \dot{p}\bar{p})\xi \right] dr,\end{aligned}$$

which justifies the assertion.

Theorem 4.1. ($n = 4$) *The solutions to the Euler–Lagrange equation associated with the energy ϕ over the space $\pi[a, b]$ are given by the pair of the quaternionic exponentials (p, q)*

$$p(r) = \mathbf{Exp}\left(\frac{1}{4} \left[\frac{1}{a^4} - \frac{1}{r^4} \right] \alpha \right), \tag{4.4}$$

$$q(r) = \mathbf{Exp}\left(\frac{1}{4} \left[\frac{1}{a^4} - \frac{1}{r^4} \right] \beta \right), \tag{4.5}$$

where $\alpha, \beta \in \mathbb{H}_0$ are constant and have the form

$$(\alpha, \beta) = (\rho_\alpha \omega_\alpha, \rho_\beta \omega_\beta),$$

where

$$\rho_\alpha = 4\pi \frac{a^4 b^4}{(b^4 - a^4)} m_\alpha$$

and

$$\rho_\beta = 4\pi \frac{a^4 b^4}{(b^4 - a^4)} m_\beta,$$

with $m_\alpha, m_\beta \in \mathbb{Z}$ *such that* $m_\beta - m_\alpha$ *is even while* $\omega_\alpha, \omega_\beta \in \mathbb{H}_0$ *and* $|\omega_\alpha| = 1$, $|\omega_\beta| = 1$ *(i.e.* ω_α, ω_β *are unit pure quaternions).*

Proof. As the pair (p, q) is a solution to (4.3) there exist $\alpha, \beta \in \mathbb{H}_0$ such that

$$\left(r^5 \frac{dp}{dr} \bar{p}, r^5 \frac{dq}{dr} \bar{q} \right) = (\alpha, \beta)$$

and consequently

$$\left(\frac{dp}{dr}, \frac{dq}{dr} \right) = \frac{1}{r^5} (\alpha p, \beta q).$$

Integrating the above further gives

$$p(r) = \mathbf{Exp}(-4^{-1} r^{-4} \alpha) p_\circ,$$
$$q(r) = \mathbf{Exp}(-4^{-1} r^{-4} \beta) q_\circ,$$

where $(p_\circ, q_\circ) \in \mathbf{Sp}(1) \times \mathbf{Sp}(1)$. Referring to the end-point conditions $(p(a), q(a)) = (1, 1)$ it is evident that

$$(p(a), q(a)) = (1,1) \Leftrightarrow \begin{cases} \mathbf{Exp}(-4^{-1} a^{-4} \alpha) p_\circ = 1 \\ \mathbf{Exp}(-4^{-1} a^{-4} \beta) q_\circ = 1 \end{cases}$$
$$\Leftrightarrow \begin{cases} p_\circ = \mathbf{Exp}(4^{-1} a^{-4} \alpha) \\ q_\circ = \mathbf{Exp}(4^{-1} a^{-4} \beta). \end{cases}$$

In a similar way for the end-point condition $(q(a), q(b)) = \pm(1, 1)$ upon substitution from above we have

$$(p(b), q(b)) = \pm(1,1) \Leftrightarrow \begin{cases} \mathbf{Exp}(s\alpha) q_\circ = 1, \\ \mathbf{Exp}(s\beta) p_\circ = 1, \\ \text{or,} \\ \mathbf{Exp}(s\alpha) q_\circ = -1, \\ \mathbf{Exp}(s\beta) p_\circ = -1, \end{cases}$$

where for the sake of brevity and convenience we have set

$$s := \frac{1}{4} \frac{(b^4 - a^4)}{a^4 b^4}.$$

Therefore, to conclude it is enough to observe that

$$\begin{bmatrix} \mathbf{Exp}(s\alpha), \\ \mathbf{Exp}(s\beta) \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \mathbf{Exp}(s\rho_\alpha \omega_\alpha), \\ \mathbf{Exp}(s\rho_\beta \omega_\beta) \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} \cos(s\rho_\alpha) + \omega_\alpha \sin(s\rho_\alpha) \\ \cos(s\rho_\beta) + \omega_\beta \sin(s\rho_\beta) \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} s\rho_\alpha \in \pi\mathbb{Z} \\ s\rho_\beta \in \pi\mathbb{Z} \\ s(\rho_\beta - \rho_\alpha) \in 2\pi\mathbb{Z} \end{cases}$$

$$\Leftrightarrow \begin{cases} \rho_\alpha \in s^{-1}\pi\mathbb{Z} \\ \rho_\beta \in s^{-1}\pi\mathbb{Z} \\ \rho_\beta - \rho_\alpha \in 2s^{-1}\pi\mathbb{Z} \end{cases},$$

which is the required conclusion. □

The immediate aim next is to find the $\mathbf{SO}(4)$-valued matrix $\mathbf{Q}_{p,q}$ associated with the pair (p, q) from the above theorem as described by (A.3). To this end we proceed by introducing the functions

$$\begin{Bmatrix} g_1(r) \\ g_2(r) \end{Bmatrix} := \frac{1}{4}\frac{(r^4 - a^4)}{a^4 r^4} \times \begin{Bmatrix} \rho_\alpha - \rho_\beta \\ \rho_\alpha + \rho_\beta \end{Bmatrix},$$

with $r \in [a, b]$. Then, writing $\alpha = \rho_\alpha\omega_\alpha$ and $\beta = \rho_\beta\omega_\beta$ as in the theorem with $\omega_\alpha, \omega_\beta$ being unit pure quaternions we have

$$p(r) = \mathbf{Exp}\left(\frac{1}{2}[g_1 + g_2](r)\omega_\alpha\right), \tag{4.6}$$

$$q(r) = \mathbf{Exp}\left(\frac{1}{2}[g_2 - g_1](r)\omega_\beta\right), \tag{4.7}$$

where with the aid of (3.6) this can also be expressed as

$$\begin{cases} p(r) = \cos\left(\frac{1}{2}[g_1 + g_2]\right) + \omega_\alpha \sin\left(\frac{1}{2}[g_1 + g_2]\right), \\ q(r) = \cos\left(\frac{1}{2}[g_2 - g_1]\right) + \omega_\beta \sin\left(\frac{1}{2}[g_2 - g_1]\right). \end{cases}$$

Next, in view of $\omega_\alpha, \omega_\beta$ being unit pure quaternions there exist $\xi, \zeta \in \mathbf{Sp}(1)$ such that

$$\overline{\omega_\alpha} = \xi i \bar{\xi}$$

and in a similar way

$$\overline{\omega_\beta} = \zeta i \bar{\zeta}.$$

As a result by combining the above we can write

$$\begin{aligned} (p, q) &= \left(\mathbf{Exp}\left(\frac{1}{2}[g_1 + g_2]\omega_\alpha\right), \mathbf{Exp}\left(\frac{1}{2}[g_2 - g_1]\omega_\beta\right)\right) \\ &= \left(\xi\left(\cos\frac{1}{2}[g_1 + g_2] - i \sin\frac{1}{2}[g_1 + g_2]\right)\bar{\xi}, \zeta\right. \\ &\quad \left. \times \left(\cos\frac{1}{2}[g_2 - g_1] - i \sin\frac{1}{2}[g_2 - g_1]\right)\bar{\zeta}\right) \end{aligned}$$

$$= \left(\xi \mathbf{Exp}\left(-\frac{1}{2}[g_1+g_2]i\right)\bar{\xi}, \zeta \mathbf{Exp}\left(-\frac{1}{2}[g_2-g_1]i\right)\bar{\zeta}\right)$$
$$=: (\xi\eta_1\bar{\xi}, \zeta\eta_2\bar{\zeta}).$$

Therefore, the $\mathbf{SO}(4)$-valued map $\mathbf{Q}_{p,q}$ associated with the solution pair (p,q) given in Theorem 4.1 can be written in the *diagonalized* form

$$\mathbf{Q}_{p,q} = \mathbf{P}\mathbf{Q}_{\eta_1,\eta_2}\mathbf{P}^t, \tag{4.8}$$

where $\mathbf{P} = \mathbf{Q}_{\xi,\zeta} \in \mathbf{SO}(4)$ is constant and the $\mathbf{SO}(4)$-valued map $\mathbf{Q}_{\eta_1,\eta_2}$ [with $\eta_1(r) = \mathbf{Exp}(-\frac{1}{2}[g_1+g_2](r)i)$ and $\eta_2(r) = \mathbf{Exp}(-\frac{1}{2}[g_2-g_1](r)i)$] has the specific form

$$[\mathbf{Q}_{\eta_1,\eta_2}] =: \mathrm{diag}(\mathbf{R}[g_1], \mathbf{R}[g_2])$$
$$= \begin{bmatrix} \cos g_1 & \sin g_1 & 0 & 0 \\ -\sin g_1 & \cos g_1 & 0 & 0 \\ 0 & 0 & \cos g_2 & \sin g_2 \\ 0 & 0 & -\sin g_2 & \cos g_2 \end{bmatrix} \tag{4.9}$$

corresponding to two sets of independent twists.

Proposition 4.1. *Let* $(p,q) = (p,q)[r;a,b,\alpha,\beta]$ *denote the solution pair to the Euler–Lagrange equation associated with the energy* ϕ *over the space* $\pi[a,b]$. (*See Theorem* 4.1.) *Then, with the notation used in the theorem we have that*

$$\phi[(p,q)] = 4\pi^2 \frac{a^4b^4}{(b^4-a^4)}[m_\alpha^2 + m_\beta^2].$$

Proof. Utilizing the explicit expression for the solution pair from the theorem above a direct calculation gives

$$\phi[(p,q)] = \int_a^b (|\dot{p}|^2 + |\dot{q}|^2) r^5\, dr$$
$$= \int_a^b \frac{1}{r^5}(|\alpha p|^2 + |\beta q|^2)\, dr = \frac{1}{4}\frac{(b^4-a^4)}{a^4b^4}(|\alpha|^2 + |\beta|^2).$$

As a result in view of $\alpha = \alpha(m_\alpha, \omega_\alpha)$ and $\beta = \beta(m_\beta, \omega_\beta)$ we can write

$$\phi[(p,q)] = \frac{1}{4}\frac{(b^4-a^4)}{a^4b^4}(|\alpha|^2 + |\beta|^2)$$
$$= \frac{1}{4}\frac{(b^4-a^4)}{a^4b^4}\left[\left(4\pi\frac{a^4b^4}{(b^4-a^4)}m_\alpha\right)^2 + \left(4\pi\frac{a^4b^4}{(b^4-a^4)}m_\beta\right)^2\right]$$
$$= 4\pi^2\frac{a^4b^4}{(b^4-a^4)}[m_\alpha^2 + m_\beta^2],$$

which is the required conclusion.[e] □

[e]It is convenient to set $m_1 = (m_\alpha - m_\beta)/2$ and $m_2 = (m_\alpha + m_\beta)/2$. Then, $m_1 \in \mathbb{Z}$, $m_2 \in \mathbb{Z}$ if and only if $m_\alpha \in \mathbb{Z}$, $m_\beta \in \mathbb{Z}$ and $m_\beta - m_\alpha \in 2\mathbb{Z}$ (see Theorem 4.1).

Now, for the *generalized* twist associated with the solution pair (p, q) from Theorem 4.1 we have the corresponding Dirichlet energy given by

$$\begin{aligned} \mathbb{F}[u; \mathbf{X}] &= \pi^2 \left[(b^4 - a^4) + 4\pi^2 \frac{a^4 b^4}{(b^4 - a^4)} [m_\alpha^2 + m_\beta^2] \right] \\ &= \pi^2 \left[(b^4 - a^4) + 8\pi^2 \frac{a^4 b^4}{(b^4 - a^4)} [m_1^2 + m_2^2] \right]. \end{aligned} \tag{4.10}$$

[It is instructive to compare the above energy as well as the functions g_1, g_2 with those expressed in [10, p. 203].]

5. Generalized Twists as Classical Solutions to the Euler–Lagrange Equation

In this section we *characterize* all those stationary loops $\mathbf{Q} = \mathbf{Q}_q$ $(n = 3)$ and $\mathbf{Q} = \mathbf{Q}_{p,q}$ $(n = 4)$ whose resulting *generalized* twist furnishes a solution to the system of Euler–Lagrange equations associated with the energy $\mathbb{F}$ over $\mathcal{A}(\mathbf{X})$. To this end we begin by first clarifying the notion of a solution.

Definition 5.1 (Classical Solution). A pair $(u, \mathfrak{p})$ is said to be a *classical* solution to the Euler–Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}(\mathbf{X})$ if and only if the following hold[f]:

[**1**] $u \in \mathbf{C}^2(\mathbf{X}, \mathbb{R}^n) \cap \mathbf{C}(\bar{\mathbf{X}}, \mathbb{R}^n)$,
[**2**] $\mathfrak{p} \in \mathbf{C}^1(\mathbf{X}) \cap \mathbf{C}(\bar{\mathbf{X}})$,
[**3**] $(u, \mathfrak{p})$ satisfy the *system* of equations

$$\begin{cases} [\mathrm{cof}\nabla u]^{-1} \Delta u = \nabla \mathfrak{p} & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u = x & \text{on } \partial \mathbf{X}. \end{cases}$$

Remark 5.1. As the stationary loops $\mathbf{Q} = \mathbf{Q}_q$ $(n = 3)$ and $\mathbf{Q} = \mathbf{Q}_{p,q}$ $(n = 4)$ constructed earlier [see Theorems 3.1 and 4.1] are smooth [see (3.4) and (4.4), (4.5)] it follows that the same is true of the resulting *twists*. Hence, the focus in this section will be entirely on analyzing the system of equations in [**3**] above.

Proposition 5.1. *Let u be a generalized twist with $\mathbf{Q} \in \mathbf{C}^2(]a, b[, \mathbf{SO}(n))$. Then*

$$[\mathrm{cof}\, \nabla u]^{-1} \Delta u = [(n+1)\mathbf{Q}^t \dot{\mathbf{Q}} + r\mathbf{Q}^t \ddot{\mathbf{Q}} + (n+1) r |\dot{\mathbf{Q}}\theta|^2 + r^2 \langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta \rangle]\theta.$$

Proof. Referring to Definition (2.1) it follows upon differentiating once that

$$\nabla u = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta.$$

[f] It is implicit here that the map u itself is *admissible*, i.e. $u \in \mathcal{A}$ and so this condition is not iterated in the list.

Next, using the notation $u = (u_1, u_2, \dots, u_n)$ it follows from further component-wise differentiation that

$$\begin{aligned}
\Delta u_i &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left\{ \mathbf{Q}_{ij} + r \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k \theta_j \right\} \\
&= \sum_{j=1}^{n} \left\{ \dot{\mathbf{Q}}_{ij} \theta_j + \theta_j \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k \theta_j + r \sum_{k=1}^{n} \ddot{\mathbf{Q}}_{ik} \theta_j \theta_k \theta_j \right. \\
&\quad \left. + \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} (\delta_{kj} - \theta_j \theta_k) \theta_j + \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k (1 - \theta_j \theta_j) \right\} \\
&= (n+1) \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k + r \sum_{j=1}^{n} \ddot{\mathbf{Q}}_{ij} \theta_j .
\end{aligned}$$

As this is true for $1 \le i \le n$ going back to the original *vector* notation we arrive at

$$\Delta u = [(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}]\theta.$$

Now, as a consequence of the pointwise identity $\det \nabla u = 1$ it follows that we have

$$[\mathrm{cof} \nabla u]^{-1} = [\nabla u]^t = \mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta.$$

Therefore, by combining the above we can write

$$\begin{aligned}
[\mathrm{cof}\, \nabla u]^{-1} \Delta u &= [\nabla u]^t \Delta u \\
&= [\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta]^t \times [(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}]\theta \\
&= [(n+1)\mathbf{Q}^t\dot{\mathbf{Q}} + r\mathbf{Q}^t\ddot{\mathbf{Q}} + (n+1)r\,|\dot{\mathbf{Q}}\theta|^2 + r^2 \langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta \rangle]\theta,
\end{aligned}$$

which is the required conclusion. □

5.1. *Classical solutions and stationary loops in* Spin(3)

This section is devoted to the case $n = 3$. Before proceeding further, however, we pause briefly to comment on notation.

Indeed, we identify $\mathbb{R}^3$ with $\mathbb{H}_0$ and for convenience often abuse notation and represent the vector (x_1, x_2, x_3) and the *pure* quaternion $ix_1 + jx_2 + kx_3$ by the same symbol x. Using this identification we will regard an element of $\mathbb{M}_{1\times 3}(\mathbb{H}_0)$ with one of $\mathbb{M}_{3\times 3}(\mathbb{R})$ whose columns are vectors in $\mathbb{R}^3$ representing the entries of the former in $\mathbb{H}_0$. We set $\mathbf{U} \in \mathbb{M}_{1\times 3}(\mathbb{H}_0)$ to denote

$$\mathbf{U} = [i, j, k] \tag{5.1}$$

and for $\theta \in \mathbb{S}^2$ we set

$$\mathbf{S} = \mathbf{S}(\theta) = [\theta_1, \theta_2, \theta_3]. \tag{5.2}$$

Using this notation we can write $\mathbf{Q}_q = q\mathbf{U}\bar{q}$ [see (A.1)] and thus $\dot{\mathbf{Q}}_q = \dot{q}\mathbf{U}\bar{q} + q\mathbf{U}\dot{\bar{q}}$. Hence, for the *generalized* twist u we obtain

$$\begin{aligned}\nabla u &= \mathbf{Q}_q + r\dot{\mathbf{Q}}_q\theta \otimes \theta \\ &= q\mathbf{U}\bar{q} + r(\dot{q}\theta\bar{q} + q\theta\dot{\bar{q}})\mathbf{S} \\ &= q[\mathbf{U} + r(\bar{q}\dot{q}\theta + \theta\dot{\bar{q}}q)\mathbf{S}]\bar{q},\end{aligned}$$

where in the *first* equality θ is identified with the unit vector $(\theta_1, \theta_2, \theta_3)$ whereas in the *second* and *third* equalities θ is identified with the unit pure quaternion $i\theta_1 + j\theta_2 + k\theta_3$. In a similar way, we have that

$$\begin{aligned}\Delta u &= [4\dot{\mathbf{Q}}_q + r\ddot{\mathbf{Q}}_q]\theta \\ &= 4(\dot{q}\theta\bar{q} + q\theta\dot{\bar{q}}) + r(\ddot{q}\theta\bar{q} + 2\dot{q}\theta\dot{\bar{q}} + q\theta\ddot{\bar{q}}) \\ &= q[4(\bar{q}\dot{q}\theta + \theta\dot{\bar{q}}q) + r(\bar{q}\ddot{q}\theta + 2\bar{q}\dot{q}\theta\dot{\bar{q}}q + \theta\ddot{\bar{q}}q)]\bar{q}.\end{aligned}$$

Therefore, by referring to Proposition 5.1 we can write[g]

$$\begin{aligned}[\operatorname{cof} \nabla u]^{-1}\Delta u &= [\nabla u]^t\Delta u \\ &= [4\mathbf{Q}_q^t\dot{\mathbf{Q}}_q + r\mathbf{Q}_q^t\ddot{\mathbf{Q}}_q + 4r|\dot{\mathbf{Q}}_q\theta|^2 + r^2\langle\dot{\mathbf{Q}}_q\theta, \ddot{\mathbf{Q}}_q\theta\rangle]\theta \\ &= [4(q\mathbf{U}^t\bar{q})(\dot{q}\mathbf{U}\bar{q} + q\mathbf{U}\dot{\bar{q}}) + r(q\mathbf{U}^t\bar{q})(\ddot{q}\mathbf{U}\bar{q} + 2\dot{q}\mathbf{U}\dot{\bar{q}} + q\mathbf{U}\ddot{\bar{q}}) \\ &\quad + 4r|\dot{q}\theta\bar{q} + q\theta\dot{\bar{q}}|^2 + r^2\langle\dot{q}\theta\bar{q} + q\theta\dot{\bar{q}}, \ddot{q}\theta\bar{q} + 2\dot{q}\theta\dot{\bar{q}} + q\theta\ddot{\bar{q}}\rangle]\theta.\end{aligned}$$

For any $\mathbf{Sp}(1)$-valued map q arising as a solution to the Euler–Lagrange equation associated with the energy ϕ over the space $\pi[a, b]$ the expression on the *right* can be considerably simplified as is stated below.

Proposition 5.2. *Let* $\mathbf{X} = \{x \in \mathbb{R}^3 : a < |x| < b\}$ *and* u *be the generalized twist associated with* $\mathbf{Q}_q$, *where* q *is as described in Theorem* 3.1. *Then,*

$$[\operatorname{cof} \nabla u]^{-1}\Delta u = \frac{2}{r^8}\rho_\alpha^2[\mathbf{Q}_{\omega_\alpha} - \mathbf{I}_3]x,$$

for all $x \in \mathbf{X}$.

Proof. Referring to Theorem 3.1 the solution $q = q(r)$ can be expressed by the quaternionic exponential

$$q(r) = \mathbf{Exp}\left(\frac{1}{3}\frac{(r^3 - a^3)}{a^3r^3}\alpha\right),$$

[g]Note that for $p, q \in \mathbb{H}$ we set $\langle p, q\rangle = \Re(p\bar{q})$. (This corresponds to the *usual* inner product of the vectors (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) in $\mathbb{R}^4$.)

where $\alpha \in \mathbb{H}_0$. Now, referring to the discussion prior to the proposition we have that

$$\nabla u = q\left[\mathbf{U} + \frac{1}{r^3}(\alpha\theta - \theta\alpha)\mathbf{S}\right]\bar{q},$$
$$\Delta u = q\left[-\frac{2}{r^7}(|\alpha|^2\theta + \alpha\theta\alpha)\right]\bar{q}$$

and consequently with $\alpha = \rho_\alpha\omega_\alpha$ and $\langle\alpha,\beta\rangle = \Re(\alpha\bar{\beta})$ as indicated earlier, we have

$$\begin{aligned}[\mathrm{cof}\,\nabla u]^{-1}\Delta u &= \left\{q\left[\mathbf{U}^t + \frac{1}{r^3}(\alpha\theta - \theta\alpha)\mathbf{S}^t\right]\bar{q}\right\} \times \left\{q\left[-\frac{2}{r^7}(|\alpha|^2\theta + \alpha\theta\alpha)\right]\bar{q}\right\} \\ &= -\frac{2}{r^7}\rho_\alpha^2(\omega_\alpha\theta\omega_\alpha + \theta) + \rho_\alpha^3\left\langle\frac{1}{r^3}(\omega_\alpha\theta - \theta\omega_\alpha), -\frac{2}{r^7}(\theta + \omega_\alpha\theta\omega_\alpha)\right\rangle\theta \\ &= \frac{2}{r^7}\rho_\alpha^2[\omega_\alpha\theta\bar{\omega}_\alpha - \theta],\end{aligned}$$

which is the required conclusion. □

Theorem 5.1. $(n = 3)$ *The only classical solution to the system of Euler–Lagrange equations associated with the energy* $\mathbb{F}$ *over* $\mathcal{A}(\mathbf{X})$ *in the form of generalized twist is* $u = \varphi$ *(corresponding to* $q = 1$*).*

Proof. Let u as described be a *generalized* twist and write $u(x) = q(r)x\bar{q}(r)$, where the $\mathbf{Sp}(1)$-valued map q is as in Theorem 3.1 given by

$$q(r) = \mathbf{Exp}\left(\frac{1}{3}\frac{(r^3 - a^3)}{a^3r^3}\alpha\right),$$

where $\alpha \in \mathbb{H}_0$. For the sake of a contradiction assume $\alpha \neq 0$. Then, referring to Proposition 5.2 we have (with $\alpha := \rho_\alpha\omega_\alpha$) that

$$\begin{aligned}[\mathrm{cof}\,\nabla u]^{-1}\Delta u &= \frac{2}{r^8}\rho_\alpha^2[\mathbf{Q}_{\omega_\alpha} - \mathbf{I}_3]x \\ &= \frac{2}{r^8}\rho_\alpha^2\mathbf{Q}_{\omega_\alpha}x + \nabla s,\end{aligned}$$

where $s := \frac{1}{3}\rho_\alpha^2|x|^{-6}$. As a result of this u is a classical solution if and only if the vector field v given by

$$v(x) := \frac{2}{r^8}\rho_\alpha^2\mathbf{Q}_{\omega_\alpha}x,$$

is a gradient in $\mathbf{X}$. However, we have that

$$\begin{aligned} v \text{ is a } \textit{gradient} &\Rightarrow \nabla \times v = 0 \\ &\Rightarrow \frac{16}{r^9}\rho_\alpha^2(\mathbf{Q}_{\omega_\alpha}\theta \otimes \theta - \theta \otimes \mathbf{Q}_{\omega_\alpha}\theta) = 0,\end{aligned}$$

the last identity holding for all $\theta \in \mathbb{S}^2$ and $r \in]a,b[$. (Notice that here we have used the *symmetry* of $\mathbf{Q}_{\omega_\alpha}$.) However, in view of $\mathbf{Q}_{\omega_\alpha}$ being additionally special orthogonal (hence having eigenvalues ± 1 only) we have that

$$\begin{aligned}(\mathbf{Q}_{\omega_\alpha}\theta \otimes \theta - \theta \otimes \mathbf{Q}_{\omega_\alpha}\theta) = 0 &\Rightarrow (\mathbf{Q}_{\omega_\alpha}\theta \otimes \theta)\theta = (\theta \otimes \mathbf{Q}_{\omega_\alpha}\theta)\theta \\ &\Rightarrow \mathbf{Q}_{\omega_\alpha}\theta = \langle \mathbf{Q}_{\omega_\alpha}\theta, \theta\rangle\theta \\ &\Rightarrow \langle \mathbf{Q}_{\omega_\alpha}\theta, \theta\rangle^2 = 1 \\ &\Rightarrow \mathbf{Q}_{\omega_\alpha} = \mathbf{I}_3 \\ &\Rightarrow \omega_\alpha \in \{\pm 1\},\end{aligned}$$

which is a contradiction as α is a pure quaternion. This completes the proof. □

5.2. *Classical solutions and stationary loops in* Spin(4)

This section is devoted to the case $n = 4$. Notation is similar to that described in the previous section with natural modifications corresponding to identifying $\mathbb{R}^4$ with $\mathbb{H}$. Here, we set $\mathbf{U} \in \mathbb{M}_{1\times 4}(\mathbb{H})$ to denote

$$\mathbf{U} = [1, i, j, k] \tag{5.3}$$

and for $\theta \in \mathbb{S}^3$ we set

$$\mathbf{S} = \mathbf{S}(\theta) = [\theta_1, \theta_2, \theta_3, \theta_4]. \tag{5.4}$$

Now, using this notation we can write $\mathbf{Q}_{p,q} = p\mathbf{U}\bar{q}$ [see (A.3)] and $\dot{\mathbf{Q}}_{p,q} = \dot{p}\mathbf{U}\bar{q} + p\mathbf{U}\dot{\bar{q}}$. Hence, similar to that done in the previous section for the *generalized* twist u we obtain

$$\nabla u = p[\mathbf{U} + r(\bar{p}\dot{p}\theta + \theta\dot{\bar{q}}q)\mathbf{S}]\bar{q}$$

and

$$\Delta u = p[5(\bar{p}\dot{p}\theta + \theta\dot{\bar{q}}q) + r(\bar{p}\ddot{p}\theta + 2\bar{p}\dot{p}\theta\dot{\bar{q}}q + \theta\ddot{\bar{q}}q)]\bar{q}$$

and thus

$$\begin{aligned}[\operatorname{cof}\nabla u]^{-1}\Delta u &= [\nabla u]^t\Delta u \\ &= [5\mathbf{Q}_{p,q}^t\dot{\mathbf{Q}}_{p,q} + r\mathbf{Q}_{p,q}^t\ddot{\mathbf{Q}}_{p,q} + 5r|\dot{\mathbf{Q}}_{p,q}\theta|^2 + r^2\langle\dot{\mathbf{Q}}_{p,q}\theta, \ddot{\mathbf{Q}}_{p,q}\theta\rangle]\theta \\ &= [5(p\mathbf{U}^t\bar{q})(\dot{p}\mathbf{U}\bar{q} + p\mathbf{U}\dot{\bar{q}}) + r(p\mathbf{U}^t\bar{q})(\ddot{p}\mathbf{U}\bar{q} + 2\dot{p}\mathbf{U}\dot{\bar{q}} + p\mathbf{U}\ddot{\bar{q}}) \\ &\quad + 5r|\dot{p}\theta\bar{q} + p\theta\dot{\bar{q}}|^2 + r^2\langle\dot{p}\theta\bar{q} + p\theta\dot{\bar{q}}, \ddot{p}\theta\bar{q} + 2\dot{p}\theta\dot{\bar{q}} + p\theta\ddot{\bar{q}}\rangle]\theta.\end{aligned}$$

Proposition 5.3. *Let* $\mathbf{X} = \{x \in \mathbb{R}^4 : a < |x| < b\}$ *and* u *be the generalized twist associated with* $\mathbf{Q}_{p,q}$ *where the pair* (p,q) *is as described in Theorem* 4.1. *Then,*

$$[\operatorname{cof}\nabla u]^{-1}\Delta u = \frac{1}{r^{10}}[2\rho_\alpha\rho_\beta\mathbf{Q}_{\omega_\alpha,\omega_\beta} - (\rho_\alpha^2 + \rho_\beta^2)\mathbf{I}_4]x,$$

for all $x \in \mathbf{X}$.

Proof. Referring to Theorem 4.1 the solution pair (p, q) can be expressed by the pair of quaternionic exponentials

$$p(r) = \mathbf{Exp}\left(\frac{1}{4}\frac{(r^4 - a^4)}{a^4 r^4}\alpha\right),$$

$$q(r) = \mathbf{Exp}\left(\frac{1}{4}\frac{(r^4 - a^4)}{a^4 r^4}\beta\right),$$

where $\alpha, \beta \in \mathbb{H}_0$. Now, a straightforward calculation similar to that in the case $n = 3$ gives

$$\nabla u = p\left[\mathbf{U} + \frac{1}{r^4}(\alpha\theta + \theta\bar{\beta})\mathbf{S}\right]\bar{q},$$

$$\Delta u = p\left[-\frac{1}{r^9}(|\alpha|^2 + |\beta|^2)\theta + \frac{2}{r^9}\alpha\theta\bar{\beta}\right]\bar{q}$$

and consequently

$$\begin{aligned}
[\operatorname{cof} \nabla u]^{-1}\Delta u &= \left\{p\left[\mathbf{U}^t + \frac{1}{r^4}(\alpha\theta + \theta\bar{\beta})\mathbf{S}^t\right]\bar{q}\right\} \\
&\quad \times \left\{p\left[-\frac{1}{r^9}(|\alpha|^2 + |\beta|^2)\theta + \frac{2}{r^9}\alpha\theta\bar{\beta}\right]\bar{q}\right\} \\
&= -\frac{1}{r^9}(|\alpha|^2 + |\beta|^2)\theta + \frac{2}{r^9}\alpha\theta\bar{\beta} \\
&\quad + \left\langle \frac{1}{r^4}(\alpha\theta + \theta\bar{\beta}), -\frac{1}{r^9}(|\alpha|^2 + |\beta|^2)\theta + \frac{2}{r^9}\alpha\theta\bar{\beta}\right\rangle \theta \\
&= -\frac{1}{r^9}(|\alpha|^2 + |\beta|^2)\theta + \frac{2}{r^9}\alpha\theta\bar{\beta},
\end{aligned}$$

which is the required conclusion. □

Theorem 5.2. ($n = 4$) *Every classical solution to the system of Euler–Lagrange equations associated with the energy* $\mathbb{F}$ *over* $\mathcal{A}(\mathbf{X})$ *in the form of a generalized twist corresponds only to one of the pairs*

$$p(r) = \mathbf{Exp}\left(\frac{1}{4}\frac{(r^4 - a^4)}{a^4 r^4}\alpha\right),$$

$$q(r) = 1,$$

or

$$p(r) = 1,$$

$$q(r) = \mathbf{Exp}\left(\frac{1}{4}\frac{(r^4 - a^4)}{a^4 r^4}\beta\right).$$

In other words the $\mathbf{SO}(4)$-valued map $\mathbf{Q}_{p,q}$ corresponds to a solution pair (p, q) from Theorem 4.1 with either $\alpha = 0$ or $\beta = 0$.

Proof. Let u as described be a *generalized* twist and write $u(x) = p(r)x\bar{q}(r)$, where the $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$-valued map (p, q) is as in Theorem 4.1 given by

$$p(r) = \mathbf{Exp}\left(\frac{1}{4}\frac{(r^4 - a^4)}{a^4 r^4}\alpha\right),$$

$$q(r) = \mathbf{Exp}\left(\frac{1}{4}\frac{(r^4 - a^4)}{a^4 r^4}\beta\right),$$

where $\alpha, \beta \in \mathbb{H}_0$. For the sake of a contradiction assume neither $\alpha = 0$ nor $\beta = 0$. Then, referring to Proposition 5.3 we have (with $\alpha = \rho_\alpha \omega_\alpha$ and $\beta = \rho_\beta \omega_\beta$) that

$$\begin{aligned}[\mathrm{cof}\,\nabla u]^{-1}\Delta u &= \frac{1}{r^{10}}[2\rho_\alpha\rho_\beta \mathbf{Q}_{\omega_\alpha,\omega_\beta} - (\rho_\alpha^2 + \rho_\beta^2)\mathbf{I}_4]x \\ &= \frac{2}{r^{10}}\rho_\alpha\rho_\beta \mathbf{Q}_{\omega_\alpha,\omega_\beta} x + \nabla s,\end{aligned}$$

where $s = 8^{-1}(\rho_\alpha^2 + \rho_\beta^2)|x|^{-8}$. As a result this u is a classical solution if and only if the vector field v given by

$$v(x) := \frac{2}{r^{10}}\rho_\alpha\rho_\beta \mathbf{Q}_{\omega_\alpha,\omega_\beta} x,$$

is a gradient in $\mathbf{X}$. However, we have that

$$\begin{aligned}v \text{ is a } \textit{gradient} &\Rightarrow \mathbf{curl}(v) = 0 \\ &\Rightarrow \frac{20}{r^{11}}\rho_\alpha\rho_\beta(\mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta \otimes \theta - \theta \otimes \mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta) = 0,\end{aligned}$$

the last identity holding for all $\theta \in \mathbb{S}^3$ and $r \in]a, b[$. (Notice that here we have again used the *symmetry* of $\mathbf{Q}_{\omega_\alpha,\omega_\beta}$.) However, in view of $\mathbf{Q}_{\omega_\alpha,\omega_\beta}$ being additionally special orthogonal (hence having eigenvalues ± 1 only) we have that

$$\begin{aligned}(\mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta \otimes \theta - \theta \otimes \mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta) = 0 &\Rightarrow (\mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta \otimes \theta)\theta = (\theta \otimes \mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta)\theta \\ &\Rightarrow \mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta = \langle \mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta, \theta\rangle\theta \\ &\Rightarrow \langle \mathbf{Q}_{\omega_\alpha,\omega_\beta}\theta, \theta\rangle^2 = 1 \\ &\Rightarrow \mathbf{Q}_{\omega_\alpha,\omega_\beta} = \pm\mathbf{I}_4 \\ &\Rightarrow (\omega_\alpha, \omega_\beta) \in \{\pm(1,1), \pm(1,-1)\},\end{aligned}$$

which is a contradiction as α and β are both pure quaternions. Hence, the proof is complete. □

We end the section by describing more specifically the $\mathbf{SO}(4)$-valued map $\mathbf{Q}_{p,q}$ associated with the pair (p, q) in Theorem 5.2. To this end for $r \in [a, b]$ let

$$g(r) := 2\pi m \frac{a^4 b^4}{(b^4 - a^4)}\left(\frac{1}{a^4} - \frac{1}{r^4}\right), \tag{5.5}$$

where $m \in \mathbb{Z}$. Consider *first* the case $q = 1$ (i.e. $\beta = 0$). Then, referring to (4.6) it is plain that here the twist loop $\mathbf{Q}_{p,q}$ can be represented in the form

$$\mathbf{Q}_{p,q}(r) = \mathbf{P}\,\mathrm{diag}(\mathbf{R}[g](r), \mathbf{R}[g](r))\mathbf{P}^t,$$

where $\mathbf{P} \in \mathbf{SO}(4)$ is arbitrary but *fixed*. The case $p = 1$ (i.e. $\alpha = 0$) is similar. Indeed, here, again by referring to (4.6) we have that

$$\mathbf{Q}_{p,q}(r) = \mathbf{P}\,\mathrm{diag}(\mathbf{R}[-g](r), \mathbf{R}[g](r))\mathbf{P}^t.$$

Thus, *summarizing*, a twist loop $\mathbf{Q}_{p,q}$ corresponding to a solution pair (p, q) to the Euler–Lagrange equation associated with the energy $\mathbb{E}$ over the space $\mathcal{E}$ [as expressed by (4.8) and (4.9)] will give rise to a *classical* solution in the form of a *generalized* twist if and only if either $g_1 = g_2$ [with the common value being g as in (5.5)] or else $g_1 = -g_2$ [with one of the two being g, again, as in (5.5)].[h]

Appendix A

In this section we gather together some of the basic results and calculations regarding the double covering $\gamma : \mathbf{Spin}(n) \to \mathbf{SO}(n)$.

The algebra of quaternions is denoted by $\mathbb{H}$. For $q \in \mathbb{H}$, $\Re q$ and $\Im q$ represent the *real* and *imaginary* parts of q. If $\Re q = 0$ then q is said to be *purely* imaginary (or *pure*). The *set* of all pure quaternions is denoted by $\mathbb{H}_0$. Note in particular that the following are equivalent:

[**1**] $\Re q = 0$,
[**2**] $\Im q = q$,
[**3**] $q + \bar{q} = 0$,
[**4**] $q^2 = -|q|^2$,
[**5**] $iqi + jqj + kqk = q$.

The case $n = 3$. For $q \in \mathbf{Sp}(1)$ (*fixed*) consider the linear map $\mathbf{Q}_q : \mathbb{H}_0 \to \mathbb{H}_0$ defined via action by conjugation, i.e. for $x \in \mathbb{H}_0$,

$$\mathbf{Q}_q x = qx\bar{q}. \tag{A.1}$$

Then, $\mathbf{Q}_q$ is well-defined and upon identifying $\mathbb{H}_0$ with $\mathbb{R}^3$ it can be represented by the *special* orthogonal matrix[i]

$$[\mathbf{Q}_q] = \begin{bmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2q_2q_3 - 2q_1q_4 & 2q_1q_3 + 2q_2q_4 \\ 2q_2q_3 + 2q_1q_4 & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2q_3q_4 - 2q_1q_2 \\ 2q_2q_4 - 2q_1q_3 & 2q_1q_2 + 2q_3q_4 & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{bmatrix}.$$

[h]Notice that in the notation used in (4.10) the *first* case $q = 1$ corresponds to $m_\beta = 0$ giving $m_1 = m_2$ [with the common value being $m = m_\alpha/2$ as in (5.5)] and the *second* case $p = 1$ corresponds to $m_\alpha = 0$ giving $m_1 = -m_2$ [with one of the two being $m = m_\beta/2$, again, as in (5.5)].

[i]Note that the columns of $[\mathbf{Q}_q]$ are the pure quaternions $qi\bar{q}$, $qj\bar{q}$ and $qk\bar{q}$ viewed as vectors in $\mathbb{R}^3$.

As a result of the above representation and for the sake of future reference note that here we have

[**1**] if $\Im q = 0$ (i.e. $q = \pm 1$) then $[\mathbf{Q}_q] = \mathbf{I}_3$,
[**2**] if $\Re q = 0$ (i.e. $q_1 = 0$) then $[\mathbf{Q}_q]$ is *symmetric.*

Lemma A.1. *Let* $q \in \mathbf{C}^1([a,b], \mathbf{Sp}(1))$ *and* $\mathbf{Q}_q = \mathbf{Q}_q[r]$ *be as described above. Then,*

$$|\dot{\mathbf{Q}}_q|^2 = 8|\dot{q}|^2, \tag{A.2}$$

where $\dot{\mathbf{Q}}_q = \frac{d}{dr}\mathbf{Q}_q$ *and* $\dot{q} = \frac{d}{dr}q$.

Proof. Referring to (A.1) and the matrix representation of $\mathbf{Q}_q$ we can write

$$\begin{aligned}|\dot{\mathbf{Q}}_q|^2 &= \left|\frac{d}{dr}[qi\bar{q}, qj\bar{q}, qk\bar{q}]\right|^2 \\ &= |[\dot{q}i\bar{q} + qi\dot{\bar{q}}, \dot{q}j\bar{q} + qj\dot{\bar{q}}, \dot{q}k\bar{q} + qk\dot{\bar{q}}]|^2 \\ &= |\dot{q}i\bar{q} + qi\dot{\bar{q}}|^2 + |\dot{q}j\bar{q} + qj\dot{\bar{q}}|^2 + |\dot{q}k\bar{q} + qk\dot{\bar{q}}|^2.\end{aligned}$$

Now, by expanding each of the three terms in the above identity separately we get

$$\begin{aligned}|\dot{q}i\bar{q} + qi\dot{\bar{q}}|^2 &= (\dot{q}i\bar{q} + qi\dot{\bar{q}})\overline{(\dot{q}i\bar{q} + qi\dot{\bar{q}})} \\ &= |\dot{q}|^2 - (\dot{q}i\bar{q}\dot{q}i\bar{q} + qi\dot{\bar{q}}qi\dot{\bar{q}}) + |\dot{q}|^2\end{aligned}$$

with similar expressions for the *second* and *third* terms, respectively. Thus, upon substitution we can write

$$\begin{aligned}|\dot{\mathbf{Q}}_q|^2 &= |\dot{q}i\bar{q} + qi\dot{\bar{q}}|^2 + |\dot{q}j\bar{q} + qj\dot{\bar{q}}|^2 + |\dot{q}k\bar{q} + qk\dot{\bar{q}}|^2 \\ &= |\dot{q}|^2 - (\dot{q}i\bar{q}\dot{q}i\bar{q} + qi\dot{\bar{q}}qi\dot{\bar{q}}) + |\dot{q}|^2 \\ &\quad + |\dot{q}|^2 - (\dot{q}j\bar{q}\dot{q}j\bar{q} + qj\dot{\bar{q}}qj\dot{\bar{q}}) + |\dot{q}|^2 \\ &\quad + |\dot{q}|^2 - (\dot{q}k\bar{q}\dot{q}k\bar{q} + qk\dot{\bar{q}}qk\dot{\bar{q}}) + |\dot{q}|^2 \\ &= 8|\dot{q}|^2,\end{aligned}$$

where in obtaining the last equality we have used the fact that the quantities $\bar{q}\dot{q}$ and $\dot{\bar{q}}q$ are both *pure* quaternionic-valued and hence

$$\begin{aligned}\bar{q}\dot{q} &= i\bar{q}\dot{q}i + j\bar{q}\dot{q}j + k\bar{q}\dot{q}k, \\ \dot{\bar{q}}q &= i\dot{\bar{q}}qi + j\dot{\bar{q}}qj + k\dot{\bar{q}}qk.\end{aligned}$$

The proof is thus complete. □

The case $n = 4$. For a given pair $(p,q) \in \mathbf{Sp}(1) \times \mathbf{Sp}(1)$ consider the linear map $\mathbf{Q}_{p,q} : \mathbb{H} \to \mathbb{H}$ defined for $x \in \mathbb{H}$ via

$$\mathbf{Q}_{p,q}x = px\bar{q}. \tag{A.3}$$

Then, $\mathbf{Q}_{p,q}$ is well-defined and similar to that described for $n = 3$ identifying now $\mathbb{H}$ with $\mathbb{R}^4$ it can be represented by the *special* orthogonal matrix[j]

$$[\mathbf{Q}_{p,q}] = \begin{bmatrix} p_1q_1+p_2q_2+p_3q_3+p_4q_4 & p_1q_2-p_2q_1-p_3q_4+p_4q_3 \\ -p_1q_2+p_2q_1-p_3q_4+p_4q_3 & p_1q_1+p_2q_2-p_3q_3-p_4q_4 \\ -p_1q_3+p_2q_4+p_3q_1-p_4q_2 & p_1q_4+p_2q_3+p_3q_2+p_4q_1 \\ -p_1q_4-p_2q_3+p_3q_2+p_4q_1 & -p_1q_3+p_2q_4-p_3q_1+p_4q_2 \end{bmatrix.$$

$$\begin{matrix} p_1q_3+p_2q_4-p_3q_1-p_4q_2 & p_1q_4-p_2q_3+p_3q_2-p_4q_1 \\ -p_1q_4+p_2q_3+p_3q_2-p_4q_1 & p_1q_3+p_2q_4+p_3q_1+p_4q_2 \\ p_1q_1-p_2q_2+p_3q_3-p_4q_4 & -p_1q_2-p_2q_1+p_3q_4+p_4q_3 \\ p_1q_2+p_2q_1+p_3q_4+p_4q_3 & p_1q_1-p_2q_2-p_3q_3+p_4q_4 \end{matrix}\right].$$

As a result of the above matrix representation and for the sake of future reference note that here we have

[**1**] if $(p,q) \in \{\pm(1,1)\}$ then $\mathbf{Q}_{p,q} = \mathbf{I}_4$,
[**2**] if $(p,q) \in \{\pm(1,-1)\}$ then $\mathbf{Q}_{p,q} = -\mathbf{I}_4$,
[**3**] if $\Re p = 0$ and $\Re q = 0$ then $\mathbf{Q}_{p,q}$ is *symmetric*.

Lemma A.2. *Let* $(p,q) \in \mathbf{C}^1([a,b], \mathbf{Sp}(1) \times \mathbf{Sp}(1))$ *and* $\mathbf{Q}_{p,q} = \mathbf{Q}_{p,q}[r]$ *be as described above. Then,*

$$|\dot{\mathbf{Q}}_{p,q}|^2 = 4(|\dot{p}|^2 + |\dot{q}|^2).$$

Here

$$\dot{\mathbf{Q}}_{p,q} = \frac{d}{dr}\mathbf{Q}_{p,q}, \quad \dot{p} = \frac{d}{dr}p \quad \text{and} \quad \dot{q} = \frac{d}{dr}q.$$

Proof. Again by referring to (A.3) and the matrix representation of $\mathbf{Q}_{p,q}$ we can write

$$\begin{aligned} |\dot{\mathbf{Q}}_{p,q}|^2 &= \left| \frac{d}{dr}[p\bar{q}, pi\bar{q}, pj\bar{q}, pk\bar{q}] \right|^2 \\ &= |[\dot{p}\bar{q} + p\dot{\bar{q}}, \dot{p}i\bar{q} + pi\dot{\bar{q}}, \dot{p}j\bar{q} + pj\dot{\bar{q}}, \dot{p}k\bar{q} + pk\dot{\bar{q}}]|^2 \\ &= |\dot{p}\bar{q} + p\dot{\bar{q}}|^2 + |\dot{p}i\bar{q} + pi\dot{\bar{q}}|^2 + |\dot{p}j\bar{q} + pj\dot{\bar{q}}|^2 + |\dot{p}k\bar{q} + pk\dot{\bar{q}}|^2. \end{aligned}$$

By expanding each of the four terms in the above identity separately we have that

$$\begin{aligned} |\dot{p}\bar{q} + p\dot{\bar{q}}|^2 &= (\dot{p}\bar{q} + p\dot{\bar{q}})\overline{(\dot{p}\bar{q} + p\dot{\bar{q}})} \\ &= |\dot{p}|^2 + \dot{p}\bar{q}\dot{q}\bar{p} + p\dot{\bar{q}}q\dot{\bar{p}} + |\dot{q}|^2 \end{aligned}$$

[j]Here the columns of $[\mathbf{Q}_{p,q}]$ are the quaternions $p\bar{q}$, $pi\bar{q}$, $pj\bar{q}$ and $pk\bar{q}$, respectively.

while

$$|\dot{p}i\bar{q} + pi\dot{\bar{q}}|^2 = (\dot{p}i\bar{q} + pi\dot{\bar{q}})\overline{(\dot{p}i\bar{q} + pi\dot{\bar{q}})}$$
$$= |\dot{p}|^2 - (\dot{p}i\bar{q}\dot{q}i\bar{p} + pi\dot{\bar{q}}qi\dot{\bar{p}}) + |\dot{q}|^2$$

with similar expressions for the *third* and *fourth* terms, respectively. Thus, upon substitution we get

$$\begin{aligned}|\dot{\mathbf{Q}}_{p,q}|^2 &= |\dot{p}\bar{q} + p\dot{\bar{q}}|^2 + |\dot{p}i\bar{q} + pi\dot{\bar{q}}|^2 + |\dot{p}j\bar{q} + pj\dot{\bar{q}}|^2 + |\dot{p}k\bar{q} + pk\dot{\bar{q}}|^2 \\ &= |\dot{p}|^2 + \dot{p}\bar{q}\dot{q}\bar{p} + p\dot{\bar{q}}q\dot{\bar{p}} + |\dot{q}|^2 \\ &\quad + |\dot{p}|^2 - (\dot{p}i\bar{q}\dot{q}i\bar{p} + pi\dot{\bar{q}}qi\dot{\bar{p}}) + |\dot{q}|^2 \\ &\quad \times |\dot{p}|^2 - (\dot{p}j\bar{q}\dot{q}j\bar{p} + pj\dot{\bar{q}}qj\dot{\bar{p}}) + |\dot{q}|^2 \\ &\quad \times |\dot{p}|^2 - (\dot{p}k\bar{q}\dot{q}k\bar{p} + pk\dot{\bar{q}}qk\dot{\bar{p}}) + |\dot{q}|^2 \\ &= 4(|\dot{p}|^2 + |\dot{q}|^2),\end{aligned}$$

where again in reaching the last equality we have used the fact that the quantities $\bar{q}\dot{q}$ and $\dot{\bar{q}}q$ are both *pure* quaternionic-valued and hence,

$$\bar{q}\dot{q} = i\bar{q}\dot{q}i + j\bar{q}\dot{q}j + k\bar{q}\dot{q}k,$$
$$\dot{\bar{q}}q = i\dot{\bar{q}}qi + j\dot{\bar{q}}qj + k\dot{\bar{q}}qk.$$

This completes the proof. □

Appendix B

This section is devoted to the planar case $n = 2$. As most proofs here are direct adaptations of those presented before we shall remain brief and only highlight the main differences.

It is well-known that $\mathbf{SO}(2) \cong \mathbb{S}^1$ with $\mathbb{R}$ as the *universal* cover. Here, for given $\xi \in \mathbb{R}$ one assigns $e^{-\xi i} \in \mathbb{S}^1 \subset \mathbb{C}$ and then defines the linear transformation $\mathbf{Q}_\xi : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$\mathbf{Q}_\xi x = e^{-\xi i} x.$$

It is evident that $\mathbf{Q}_\xi$ is an orientation preserving isometry having the matrix representation

$$\begin{aligned}[\mathbf{Q}_\xi] &= [e^{-\xi i}, e^{-\xi i} i] \\ &= \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} = \mathbf{R}[\xi],\end{aligned}$$

(where $\mathbf{R}[\xi] \in \mathbf{SO}(2)$ represents the usual *rotation* matrix by angle ξ). Next, for given $\xi \in \mathbf{C}([a, b], \mathbb{R})$ one proceeds by introducing the associated (*generalized*) twist through

$$u(x) = \mathbf{Q}_\xi(r)x,$$

where $r = |x|$. It is a straightforward matter to show that $u \in \mathcal{A}$ provided that the following conditions hold:

[**1**] $\xi(a) = 0$,
[**2**] $\xi(b) = 2m\pi$,[k]
[**3**] $\xi \in W^{1,2}([a,b],\mathbb{R})$.

Moreover, an easy calculation gives

$$|\nabla u|^2 = 2 + r^2|\dot{\mathbf{Q}}_\xi|^2 = 2 + r^2\dot{\xi}^2.$$

Using the above identity we can write

$$\begin{aligned}\mathbb{F}[u;\Omega] &= \frac{1}{2}\int_\Omega |\nabla u|^2 \; dx = \frac{1}{2}\int_a^b \int_{\mathbb{S}^1} \{2 + r^2\dot{\xi}^2\} r \; d\mathcal{H}^1(\theta) dr \\ &= \pi\left[(b^2 - a^2) + \int_a^b \dot{\xi}^2 r^3 dr\right].\end{aligned}$$

We can therefore introduce the energy functional

$$\phi[\xi] := \int_a^b \dot{\xi}^2 r^3 dr,$$

over the space

$$\pi[a,b] := \left\{ \begin{aligned} &\xi(a) = 0, \\ \xi : \; &\xi(b) = 2m\pi, \\ &\xi \in W^{1,2}([a,b],\mathbb{R}), \end{aligned} \right\}.$$

Proposition B.1. *The Euler–Lagrange equations associated with the energy* ϕ *over the space* $\pi[a,b]$ *takes the form*

$$\frac{d}{dr}\left(r^3 \frac{d\xi}{dr}\right) = 0,$$

on $]a,b[$.

Proposition B.2. *Consider the Euler–Lagrange equation associated with the energy* ϕ *over the space* $\pi[a,b]$. *Then, the solution subject to the boundary conditions* $\xi(a) = 0$ *and* $\xi(b) = 2m\pi$ *is given by*

$$\xi(r) = 2m\pi \frac{a^2 b^2}{(b^2 - a^2)}\left(\frac{1}{a^2} - \frac{1}{r^2}\right),$$

where $r \in [a,b]$ *and* $m \in \mathbb{Z}$.

Theorem B.1. $(n = 2)$ *Let* $\mathbf{X} = \{x \in \mathbb{R}^2 : a < |x| < b\}$ *and consider the energy* $\mathbb{F}$ *over* $\mathcal{A}(\mathbf{X})$. *Then, every classical solution to the corresponding system of*

[k] Here $m \in \mathbb{Z}$ is any arbitrary but *fixed* integer.

Euler–Lagrange equations in the form of a generalized twist is given by [*in polar coordinates*]

$$u : (r, \theta) \mapsto (r, \theta + \xi(r)),$$

where $\xi = \xi(r)$ *is as in Proposition* B.2.

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