OPERATORS OF LAPLACE TRANSFORM TYPE AND A NEW CLASS OF HYPERGEOMETRIC COEFFICIENTS

STUART BOND AND ALI TAHERI

ABSTRACT. A differential identity on the hypergeometric function ${}_2F_1(a,b;c;z)$ unifying and extending certain spectral results on the scale of Gegenbauer and Jacobi polynomials and leading to a new class of hypergeometric related scalars $\mathsf{c}_j^m(a,b,c)$ and polynomials $\mathscr{R}_m = \mathscr{R}_m(X)$ is established. The Laplace-Beltrami operator on a compact rank one symmetric space is considered next and for operators of the Laplace transform type by invoking an operator trace relation, the Maclaurin spectral coefficients of their Schwartz kernel are fully described. Other representations as well as extensions of the differential identity to the generalised hypergeometric function ${}_pF_q(\mathbf{a};\mathbf{b};z)$ are formulated and proved.

1. Introduction

Let $(\mathscr{M}, g)$ be an N-dimensional ($N \geq 2$) compact smooth Riemannian manifold without boundary and let $\Delta = \Delta_g$ denote the Laplace-Beltrami operator on $\mathscr{M}$, given in local coordinates, by $\Delta_g = 1/\sqrt{\det g} \sum \partial_j \left(\sqrt{\det g}\, g^{jk}\partial_k\right)$.

By basic spectral theory there exists a complete orthonormal basis $(f_j : j \geq 0)$ of eigenfunctions of $-\Delta_g$ in $L^2(\mathscr{M}, dv_g)$ with a spectrum $\Sigma = \Sigma(-\Delta_g)$ consisting purely of eigenvalues. Each eigenvalue has a finite multiplicity and the spectrum can be arranged as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_j \nearrow \infty$. Thus $-\Delta_g f_j = \lambda_j f_j$ and by suitably normalising $||f_j||_{L^2(\mathscr{M})} = 1$ for all $j \geq 0$ while $(f_j, f_k)_{L^2(\mathscr{M})} = 0$ for $0 \leq j \neq k$. Now for a given function $\Phi = \Phi(X)$ in the Borel functional calculus of $-\Delta_g$ the operator $\Phi(-\Delta_g)$ has a Schwartz kernel given by the spectral sum

$$K_\Phi(x,y) = \sum_{j=0}^{\infty} \Phi(\lambda_j)\, f_j(x) f_j(y), \qquad x, y \in \mathscr{M}. \tag{1.1}$$

If $\mathscr{M}$ is a compact rank one symmetric space of a Lie group then by using the addition formula for the matrix coefficients of irreducible unitary representations the above simplifies to

$$K_\Phi(\theta) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{M})}{\mathrm{Vol}(\mathscr{M})} \Phi(\lambda_k) F_k(\theta; \mathscr{M}), \tag{1.2}$$

where $F_k = F_k(\theta; \mathscr{M})$ are the spherical functions on $\mathscr{M}$, $\lambda_k = \lambda_k(\mathscr{M})$ are the numerically *distinct* eigenvalues of $-\Delta$ on $\mathscr{M}$, $M_k = M_k(\mathscr{M})$ is the dimension of

---

Date: Received: xxxxxx; Accepted: zzzzzz.

1991 *Mathematics Subject Classification.* Primary 47F05, 47B25; Secondary 33C05, 33C20, 33C45, 47D06, 47E05, 58J35.

Key words and phrases. Operators of Laplace transform type, Schwartz kernels, Laplace-Beltrami operator, Hypergeometric functions, Maclaurin spectral functions, Symmetric spaces.

the eigenspace associated with λ_k, $\theta = \theta(x, y)$ is the distance between $x, y \in \mathscr{M}$ and $\mathrm{Vol}(\mathscr{M})$ denotes the volume of $\mathscr{M}$. (*See*, e.g., [4, 19, 23, 31, 32])

The families of compact rank one symmetric spaces of interest in this paper are the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbb{R}\mathbf{P}^n = \mathbf{S}^n/\{\pm\} = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic projective space $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/\mathbf{Sp}(n)\times\mathbf{Sp}(1)$ and the Cayley projective plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$. These spaces with the exception of the real projective space $\mathbb{R}\mathbf{P}^n$ $(n \geq 1)$ and the circle $\mathbb{S}^1$ are simply-connected where $\pi_1(\mathbb{R}\mathbf{P}^n) \cong \mathbb{Z}_2$ $n \geq 2$ and $\pi_1(\mathbb{S}^1) \cong \pi_1(\mathbb{R}\mathbf{P}^1) \cong \mathbb{Z}$. (For further discussion and results see [4, 6, 31] and [11, 12, 30, 32]).

The scale of Jacobi polynomials $\mathscr{P}_k^{(\alpha,\beta)}$ (with $k \geq 0$, $\alpha, \beta > -1$) are intertwined with the spherical functions on these symmetric spaces (for suitable α, β) and here, in the simply-connected case, (1.2) can be rewritten as [1]

$$K_\Phi(\theta) = \sum_{k=0}^{\infty} \frac{M_k^n(\mathscr{M})}{\mathrm{Vol}(\mathscr{M})} \Phi(\lambda_k^n) \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta). \tag{1.3}$$

Table 1 below presents some of the relevant spectral and geometric data for the rank one symmetric spaces described above and Table 2 gives among other things the parameters α, β for the spherical functions associated with these spaces. Now in view of the Schwartz kernel K_Φ in (1.3) being an even function of θ, subject to sufficient regularity, it admits a formal Maclaurin expansion about $\theta = 0$, given by

$$\sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j)!} \frac{\partial^{2j}}{\partial\theta^{2j}} K_\Phi \bigg|_{\theta=0} = \sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j)!} \frac{b_{2j}[\Phi]}{\mathrm{Vol}(\mathscr{M})}. \tag{1.4}$$

The Maclaurin spectral coefficients $b_{2j}[\Phi]$ defined above through the successive differentiation of the Schwartz kernel at the origin can be given an interesting trace formulation by applying the following statement, in essence, a differential-spectral identity on the Jacobi polynomials. [2]

Table 1. The spectral data for rank one symmetric spaces

$\mathscr{M}$	N	λ_k^n	$M_k^n(\mathscr{M})$	$\mathrm{Vol}(\mathscr{M})$
$\mathbb{S}^n$	n	$k(k+n-1)$	$\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}$	$\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$
$\mathbb{R}\mathbf{P}^n$	n	$2k(2k+n-1)$	$\frac{(4k+n-1)(2k+n-2)!}{(2k)!(n-1)!}$	$\frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$
$\mathbb{C}\mathbf{P}^n$	$2n$	$k(k+n)$	$\frac{2k+n}{n}\left(\frac{\Gamma(k+n)}{\Gamma(n)k!}\right)^2$	$\frac{4^n\pi^n}{n!}$
$\mathbb{H}\mathbf{P}^n$	$4n$	$k(k+2n+1)$	$\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left(\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right)^2$	$\frac{4^{2n}\pi^{2n}}{(2n+1)!}$
$\mathbf{P}^2(\mathrm{Cay})$	16	$k(k+11)$	$6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}$	$\frac{3!(4\pi)^8}{11!}$

[1]When $\mathscr{M} = \mathbb{R}\mathbf{P}^n$ it suffices to set $F_k(\theta) = \mathscr{P}_{2k}^{((n-2)/2,(n-2)/2)}(\cos\theta) = \mathscr{C}_{2k}^{(n-1)/2}(\cos\theta)$.

[2]This theorem is a special case of a more general result on hypergeometric functions appearing in Theorem 2.2.

Towards this end let $\mathsf{P}_d(X) = p_0 + \sum p_i X^i$ (summation for $1 \leq i \leq d$) be a polynomial of degree $d \geq 2$ and consider the differential operator

$$\mathscr{L}_{\mathsf{P}} = \mathsf{P}_d\,(d/d\theta) = p_0 + \sum_{i=1}^{d} p_i d^i/d\theta^i. \tag{1.5}$$

Theorem. *The action of $\mathscr{L}_{\mathsf{P}}$ as in (1.5) on the Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)}$ satisfies the following identity,*

$$\begin{aligned} \mathscr{L}_{\mathsf{P}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} &= p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \sum_{j=1}^{m} \mathsf{c}_j^m(\alpha,\beta)[\lambda_k^{\alpha,\beta}]^j \\ &= p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \mathscr{R}_m(\lambda_k^{\alpha,\beta}). \end{aligned} \tag{1.6}$$

Here $\lambda_k^{\alpha,\beta} = k(\alpha+\beta+k+1)$ are the eigenvalues of the Jacobi operator (see (A.6)), $\mathsf{c}_j^m(\alpha,\beta)$ are suitable coefficients and $\mathscr{R}_m = \mathscr{R}_m(X)$ is the m^{th} degree polynomial

$$\mathscr{R}_m(X) = \sum_{j=1}^{m} \mathsf{c}_j^m(\alpha,\beta) X^j. \tag{1.7}$$

Returning now to the discussion prior to the theorem it is seen that here the Maclaurin spectral coefficients associated with the Schwartz kernel K_Φ are given by $b_0[\Phi] = \mathrm{tr}\{\Phi(-\Delta_{\mathscr{M}})\}$, and for $j \geq 1$ by

$$b_{2j}[\Phi] = \frac{\partial^{2j}}{\partial\theta^{2j}} K_\Phi(\theta)\Big|_{\theta=0} \mathrm{Vol}(\mathscr{M}) = \mathrm{tr}\{[\mathscr{R}_j\Phi](-\Delta_{\mathscr{M}})\}, \tag{1.8}$$

where tr denotes the operator trace, in this case the operator being $[\mathscr{R}_j\Phi](-\Delta_{\mathscr{M}})$. As a particular example, $\Phi(X) = e^{-tX}$ in (1.3) gives the heat kernel on $\mathscr{M}$, with the coefficients in (1.8) corresponding to the Maclaurin heat coefficients.

In this paper we specialise to functions $\Phi = \Phi(X)$ of Laplace transform type, namely, those that for a suitable L^1-summable f are given by integral

$$\Phi(X) = \int_0^\infty f(s) e^{-Xs}\, ds, \quad X \geq 0. \tag{1.9}$$

Applying (1.8), we can write the Maclaurin coefficients for K_Φ as

$$\begin{aligned} b_{2l}[\Phi(-\Delta)] &= \int_0^\infty f(s) \sum_{k=0}^{\infty} M_k^n \sum_{j=1}^{l} \mathsf{c}_j^l [\lambda_k^n]^j e^{-\lambda_k^n s}\, ds \\ &= \int_0^\infty f(s) \left[\mathscr{R}_l\left(-\frac{d}{ds}\right)\right] \mathrm{tr}\, e^{s\Delta}\, ds = \int_0^\infty f(s) b_{2l}[e^{s\Delta}]\, ds. \end{aligned} \tag{1.10}$$

Here the polynomial $\mathscr{R}_l(X)$ is defined in (2.8) and $b_{2l}[e^{s\Delta}]$ are the Maclaurin *heat* coefficients given by $b_{2l}[e^{s\Delta}] = \mathscr{R}_l\,(-d/ds)\,\mathrm{tr}\, e^{s\Delta}$. One particular example of (1.9) is when $f(s) = f_\sigma(s) = s^{a-1}e^{-s\sigma}/\Gamma(a)$ with $\Re a > 1$ in which case for Φ one recovers the resolvent operator $\mathbf{R}_\sigma$ raised to the power a. For related discussion and applications *see* [7, 15, 25, 27, 28, 29] and the references therein.

To describe the plan of the paper, Section 2 explores and extends the differential-spectral identity (1.6) to the more general context of hypergeometric functions (see Appendix A). The main theorem here (Theorem 2.2) unifies and extends these concepts to a setting where no immediate spherical function representation or spectral interpretation of the hypergeometric function $F(a, b; c; z)$ is applicable. This leads to a new characterisation of the scalars $\mathsf{c}_j^m(a, b, c)$ and polynomials $\mathscr{R}_m$ in (1.7), reducing to the Jacobi polynomial case when $a = -k$, $b = \alpha + \beta + k + 1$, $c = \alpha + 1$ (see Table 2 below). As an application in the remaining sections we invoke these ideas along with the trace formulation of the Maclaurin spectral coefficients (1.8) to operators of Laplace transform type on a scale of compact rank one symmetric spaces to give a representation of these spectral coefficients via those of the heat semigroup and the Jacobi theta functions. Let us finish off this introduction by highlighting some important special cases of the hypergeometric function $F(a, b; c; z)$ for future reference (*cf.*, e.g., [1, 3, 16, 22]).

- The Legendre polynomial $P_k(t)$, $k \geq 0$,

$$\begin{aligned} P_k(t) &= F(-k, k+1; 1; (1-t)/2) \\ &= \frac{1}{2^k k!} \frac{d^k}{dt^k} \left[\left(t^2 - 1\right)^k \right]. \end{aligned} \tag{1.11}$$

- The Gegenbauer polynomial $\mathscr{C}_k^\nu(t)$, $\nu > -1/2$, $k \geq 0$,

$$\begin{aligned} \mathscr{C}_k^\nu(t) &= F(-k, 2\nu + k; \nu + 1/2; (1-t)/2) \\ &= \frac{(-1)^k}{2^k (\nu + 1/2)_k} \left(1 - t^2\right)^{-\nu + 1/2} \frac{d^k}{dt^k} \left[\left(1 - t^2\right)^{k + \nu - 1/2} \right]. \end{aligned} \tag{1.12}$$

- The Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)}(t)$, $k \geq 0$, $\alpha, \beta > -1$,

$$\begin{aligned} \mathscr{P}_k^{(\alpha,\beta)}(t) &= F(-k, \alpha + \beta + k + 1; \alpha + 1; (1-t)/2) \\ &= \frac{(-1)^k}{2^k (\alpha + 1)_k} (1-t)^{-\alpha} (1+t)^{-\beta} \frac{d^k}{dt^k} \left[(1-t)^\alpha (1+t)^\beta (1-t^2)^k \right]. \end{aligned} \tag{1.13}$$

- The incomplete Beta function $B(x; p, q)$,

$$\begin{aligned} B(x; p, q) &= \frac{x^p}{p} F(p, 1-q; p+q; x) \\ &= \int_0^x t^{p-1} (1-t)^{q-1} dt. \end{aligned} \tag{1.14}$$

Note in particular that in the Gegenbauer and Jacobi cases we have $\mathscr{P}_k^{(\alpha,\beta)}(1) = 1$ and $\mathscr{C}_k^\nu(1) = 1$ by the choice of normalisation.

TABLE 2. The parameter values for rank one symmetric spaces

$\mathscr{M}$	a	b	c	$-ab$	α	β
$\mathbb{S}^n$	$-k$	$k+n-1$	$n/2$	$k(k+n-1)$	$(n-2)/2$	$(n-2)/2$
$\mathbb{R}\mathbf{P}^n$	$-2k$	$2k+n-1$	$n/2$	$2k(2k+n-1)$	$(n-2)/2$	$(n-2)/2$
$\mathbb{C}\mathbf{P}^n$	$-k$	$k+n$	n	$k(k+n)$	$n-1$	0
$\mathbb{H}\mathbf{P}^n$	$-k$	$k+2n+1$	$2n$	$k(k+2n+1)$	$2n-1$	1
$\mathbf{P}^2(\mathrm{Cay})$	$-k$	$k+11$	8	$k(k+11)$	7	3

2. Hypergeometric coefficients and a combinatorial identity

In this section we present the main result which uses a combinatorial identity together with a recursive formula to describe the action of the differential operator $\mathscr{L}_{\mathsf{P}}$ on the hypergeometric function ${}_2F_1(a,b;c;z)$. This naturally leads to the introduction of a class of the polynomials $\mathscr{R}_m = \mathscr{R}_m(X)$ (with $m \geq 1$) and a set of scalars, the hypergeometric coefficients, $\mathsf{c}_j^m(a,b,c)$ (with $1 \leq j \leq m$) that play a central role in the paper.

Before stating the main theorem, we introduce some notation. For scalars $\rho_0, \dots, \rho_{j-1}$,

$$\prod_{p=0}^{j-1} \left(X + \rho_p\right) = \sum_{l=0}^{j} \mathsf{d}_{l,j} X^l, \tag{2.1}$$

where the coefficients $\mathsf{d}_{l,j} = \mathsf{d}_{l,j}(\rho_0, \dots, \rho_{j-1})$ are given by the sum of the products of the $\binom{j}{l}$ combinations of $j-l$ values ρ_p from the set $(\rho_p)_{p=0}^{j-1}$. In particular, we have

$$\mathsf{d}_{j,j} = 1, \;\; \mathsf{d}_{j-1,j} = \sum_{p=0}^{j-1} \rho_p, \;\; \mathsf{d}_{0,j} = \prod_{p=0}^{j-1} \rho_p. \tag{2.2}$$

Now we have the following lemma, which relates a product of Pochhammer symbols, which are essential to the hypergeometric function, to the scalars $\mathsf{d}_{l,j}$ introduced above.

Lemma 2.1. *With the Pochhammer symbol as defined in (A.2), the product of $(a)_j$ and $(b)_j$ can be written as a polynomial in ab, specifically,*

$$(a)_j (b)_j = \prod_{p=0}^{j-1} \left(ab + \underbrace{p(a+b+p)}_{\rho_p} \right) = \sum_{l=1}^{j} \mathsf{d}_{l,j} [ab]^l, \tag{2.3}$$

where the scalars $\mathsf{d}_{l,j} = \mathsf{d}_{l,j}(a+b)$ are defined in (2.1) by setting $\rho_p = p(a+b+p)$.

Proof. Referring to (A.2) we can write

$$(a)_j = \prod_{p=0}^{j-1} (a+p), \qquad (b)_j = \prod_{l=0}^{j-1} (b+l). \tag{2.4}$$

Thus applying this formulation to the product $(a)_j(b)_j$, we have

$$
\begin{aligned}
(a)_j(b)_j &= \prod_{k=0}^{j-1}(a+k)\prod_{l=0}^{j-1}(b+l) \\
&= \prod_{p=0}^{j-1}(a+p)(b+p) \\
&= \prod_{p=0}^{j-1}\left(ab+p(a+b+p)\right). \qquad (2.5)
\end{aligned}
$$

The conclusion now follows at once by observing (2.1). Note that in view of $\rho_0 = 0$ we have $\mathsf{d}_{0,j} = 0$ and so the sum on the right in (2.3) starts from $l = 1$. □

We can now present the main theorem, which shows the action of the differential operator $\mathscr{L}_{\mathsf{P}}$ on the hypergeometric function $F(a,b;c;z)$ and gives an explicit description of the associated coefficients $\mathsf{c}_j^m(a,b,c)$.

Theorem 2.2 (Hypergeometric coefficients)**.** *With $\mathscr{L}_{\mathsf{P}}$ the operator as in (1.5), for $|z| < 1$, and $a,b,c \in \mathbb{C}$ with $c \neq 0,-1,-2,\dots,$ the hypergeometric function $F(a,b;c;z)$ satisfies the identity*

$$
(\mathscr{L}_{\mathsf{P}}F)\left(a,b;c;\frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor d/2\rfloor} p_{2m}\sum_{j=1}^{m}\mathsf{c}_j^m(a,b,c)[-ab]^j. \qquad (2.6)
$$

The scalars $\{\mathsf{c}_j^m(a,b,c) : 1 \leq j \leq m\}$, hereafter referred to as the hypergeometric coefficients, are given explicitly by

$$
\mathsf{c}_j^m(a,b,c) = (-1)^j\sum_{i=j}^{m}(-2)^{-i}\mathsf{b}_i^m\mathsf{d}_{j,i}\prod_{p=0}^{i-1}(c+p)^{-1} \qquad (2.7)
$$

where b_i^m are as in (B.6), and $\mathsf{d}_{j,i} = \mathsf{d}_{j,i}(a+b)$ are the scalars as defined in (2.1) with $\rho_p = p(a+b+p)$.

Before stating the proof of this theorem, we introduce the m-degree polynomial $\mathscr{R}_m(X)$, defined as $\mathscr{R}_0(X) = 1$, and for $m \geq 1$

$$
\mathscr{R}_m(X) = \sum_{j=1}^{m}\mathsf{c}_j^m(a,b,c)X^j. \qquad (2.8)
$$

This lets us write the statement of (2.6) as

$$
(\mathscr{L}_{\mathsf{P}}F)\left(a,b;c;\frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor d/2\rfloor} p_{2m}\mathscr{R}_m(-ab). \qquad (2.9)
$$

Proof. We begin by noting that since $F\left(a,b;c;(1-\cos\theta)/2\right)$ is an even function of θ, evaluating its derivatives of odd order at zero will give zero. That is, when we apply $\mathscr{L}_{\mathsf{P}}$ to $F\left(a,b;c;(1-\cos\theta)/2\right)$ and evaluate at $\theta=0$, we have

$$\begin{aligned}(\mathscr{L}_{\mathsf{P}}F)\left(a,b;c;\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} &= p_0+\sum_{i=1}^{d} p_i \frac{d^i}{d\theta^i}F\left(a,b;c;\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0}\\ &= p_0+\sum_{m=1}^{\lfloor d/2\rfloor} p_{2m}\frac{d^{2m}}{d\theta^{2m}}F\left(a,b;c;\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0}. \end{aligned} \tag{2.10}$$

This allows us to use (B.4) with $f(\cos\theta)=F\left(a,b;c;(1-\cos\theta)/2\right)$ and then apply the differential identities on F from Appendix A to write,

$$\begin{aligned}\frac{d^{2m}}{d\theta^{2m}}F\left(a,b;c;\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} &= \left\{\sum_{j=1}^{m}\frac{\mathsf{b}_j^m}{(-2)^j}\frac{d^j}{dz^j}F(a,b;c;z)\right\}\Bigg|_{z=0}\\ \text{via (A.4)}\quad &= \sum_{j=1}^{m}\frac{\mathsf{b}_j^m}{(-2)^j}\frac{(a)_j(b)_j}{(c)_j}F(a+j,b+j;c+j,0)\\ &= \sum_{j=1}^{m}\frac{\mathsf{b}_j^m}{(-2)^j}\frac{(a)_j(b)_j}{(c)_j}. \end{aligned} \tag{2.11}$$

Referring to (A.2), we can write

$$\frac{\mathsf{b}_j^m}{(-2)^j(c)_j}=\frac{\mathsf{b}_j^m}{(-2)^j\prod_{p=0}^{j-1}(c+p)}=\frac{\mathsf{b}_j^m}{\mathscr{C}^j(c)}, \tag{2.12}$$

where we have set $\mathscr{C}^j(c)=(-2)^j\prod_{p=0}^{j-1}(c+p)$ for the sake of brevity. Substituting (2.12) back into (2.11), we have

$$\begin{aligned}\frac{d^{2m}}{d\theta^{2m}}F\left(a,b;c;\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} &= \sum_{j=1}^{m}\frac{\mathsf{b}_j^m}{\mathscr{C}^j(c)}(a)_j(b)_j\\ &= \sum_{j=1}^{m}\frac{\mathsf{b}_j^m}{\mathscr{C}^j(c)}\sum_{l=1}^{j}\mathsf{d}_{l,j}(ab)^l. \end{aligned} \tag{2.13}$$

Note that here we have invoked Lemma 2.1 to introduce $\mathsf{d}_{i,j}$. Now expanding the sum and isolating powers of ab enables us to rearrange the above expression as

$$\begin{aligned}\frac{d^{2m}}{d\theta^{2m}}F\left(a,b;c;\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} &= \sum_{j=1}^{m}(ab)^j\left(\sum_{i=j}^{m}\frac{\mathsf{b}_i^m}{\mathscr{C}^i(c)}\mathsf{d}_{j,i}\right)\\ &= \sum_{j=1}^{m}(-ab)^j\left((-1)^j\sum_{i=j}^{m}\frac{\mathsf{b}_i^m}{\mathscr{C}^i(c)}\mathsf{d}_{j,i}\right)\\ &= \sum_{j=1}^{m}(-ab)^j\mathsf{c}_j^m(a,b,c), \end{aligned} \tag{2.14}$$

where we have written $(-1)^j\sum_{i=j}^{m}\mathsf{b}_i^m\mathsf{d}_{j,i}[\mathscr{C}^i(c)]^{-1}=\mathsf{c}_j^m(a,b,c)$. The conclusion now follows at once upon substituting this back into (2.10). □

The first few hypergeometric coefficients can be found in Table 3 below. Using values of a, b, and c given in Table 2 yields the respective specialised coefficients for each rank one symmetric space.

TABLE 3. The first few coefficients $\mathsf{c}_j^m(a,b,c)$

j	c_j^1	c_j^2	c_j^3
1	$-\frac{1}{2c}$	$-\frac{3(a+b)-2c+1}{4c\,(c+1)}$	$-\frac{15(a+b)^2-15(a+b)c}{4c\,(c+1)\,(c+2)}$ $-\frac{15(a+b)+2\,c^2-9\,c+4}{4c\,(c+1)\,(c+2)}$
2	0	$\frac{3}{4c(c+1)}$	$\frac{45(a+b)-30\,c+15}{8\,c\,(c+1)\,(c+2)}$
3	0	0	$-\frac{15}{8\,c\,(c+1)\,(c+2)}$

3. Maclaurin spectral coefficients via Jacobi theta functions on rank one symmetric spaces

Returning to the Schwartz kernel $K_\Phi(\theta)$ from (1.3), where we described the Maclaurin coefficients $b_{2l}[\Phi]$ as in (1.8), we now specialise to functions $\Phi = \Phi(X)$ of the Laplace transform type. For a suitable function $f \in L^1$, we take Φ as

$$\Phi(X) = \int_0^\infty f(s)e^{-Xs}\,ds, \quad X \geq 0. \tag{3.1}$$

Applying the trace formulation (1.8) and taking advantage of (3.1) we can connect the Maclaurin spectral coefficients for K_Φ to those for the heat kernel by writing,

$$\begin{aligned} b_{2l}[\Phi(-\Delta)] &= \int_0^\infty f(s)\sum_{k=0}^\infty M_k^n \sum_{j=1}^l \mathsf{c}_j^l [\lambda_k^n]^j e^{-\lambda_k^n s}\,ds \\ &= \int_0^\infty f(s)\left[\mathscr{R}_l\left(-\frac{d}{ds}\right)\right]\operatorname{tr} e^{s\Delta}\,ds = \int_0^\infty f(s) b_{2l}[e^{s\Delta}]\,ds. \end{aligned} \tag{3.2}$$

Here the polynomial $\mathscr{R}_l(X)$ is defined in (2.8) and $b_{2l}[e^{s\Delta}]$ are the Maclaurin heat coefficients given by $b_{2l}[e^{s\Delta}] = \mathscr{R}_l(-d/ds)\operatorname{tr} e^{s\Delta}$. An interesting example is when $f(s) = f_\sigma(s) = s^{a-1}e^{-s\sigma}/\Gamma(a)$ with $\Re a > 1$ where one recovers the resolvent operator to the power a, that is, $\Phi(-\Delta) = \mathbf{R}_\sigma^a$. (*See* [15, 13, 28, 29] for more).

We now introduce the classical Jacobi theta functions $\vartheta_1, \vartheta_2, \vartheta_3$ of first, second, and third kind respectively for later use. These are defined for $s>0$ in turn as

$$\vartheta_1(s) = 1 + 2\sum_{j=1}^{\infty} e^{-j^2 s}, \tag{3.3}$$

$$\vartheta_2(s) = \sum_{j=0}^{\infty} (2j+1) e^{-(j+1/2)^2 s}, \tag{3.4}$$

$$\vartheta_3(s) = 2\sum_{j=1}^{\infty} j e^{-j^2 s}. \tag{3.5}$$

In this section we express the Maclaurin coefficients $b_{2l}[\Phi]$ in terms of these theta functions, first on the unit sphere $\mathbb{S}^n$, then the real projective space $\mathbb{R}\mathbf{P}^n$, then the complex projective space $\mathbb{C}\mathbf{P}^n$, and finally the quaternionic projective space $\mathbb{H}\mathbf{P}^n$. To this effect we introduce some notation that will be used throughout. We define the scalars $\mathsf{A}^n_m, \mathsf{B}^n_m, \mathsf{C}^n_m, \mathsf{D}^n_m$ respectively as the coefficients in the following polynomial expansions

$$\prod_{j=0}^{\mathscr{S}} (X^2 - j^2) = \sum_{m=0}^{\mathscr{S}} \mathsf{A}^n_m X^{2m+2}, \quad \mathscr{S} = \frac{n-3}{2}, \tag{3.6}$$

$$\prod_{j=\frac{1}{2}}^{\mathscr{S}-\frac{1}{2}} (X^2 - j^2) = \sum_{m=0}^{\mathscr{S}} \mathsf{B}^n_m X^{2m}, \quad \mathscr{S} = \frac{n-2}{2}, \tag{3.7}$$

$$\prod_{j=\frac{1}{2}}^{\mathscr{S}+\frac{1}{2}} (X^2 - j^2)^2 = \sum_{m=0}^{2\mathscr{S}+2} \mathsf{C}^n_m X^{2m}, \quad \mathscr{S} = \frac{n-3}{2}, \tag{3.8}$$

$$\prod_{j=1}^{\mathscr{S}} (X^2 - j^2)^2 = \sum_{m=0}^{2\mathscr{S}} \mathsf{D}^n_m X^{2m}, \quad \mathscr{S} = \frac{n-2}{2}. \tag{3.9}$$

Here n is a positive integer. In fact both (3.6) and (3.8) require $n \geq 3$ to be odd, and likewise (3.7) and (3.9) require $n \geq 4$ to be even. Note that the products on the left in (3.7) and (3.8) run over (non-whole) half integers, that is, starting from $j = 1/2$ they iterate by $j \to j+1$.

The first few values of A^n_m, B^n_m, C^n_m, and D^n_m can be easily calculated. Indeed, $\mathsf{A}^3_0 = 1$, $\mathsf{A}^5_0 = -1$, $\mathsf{A}^5_1 = 1$. Similarly, we have $\mathsf{B}^4_0 = -1/4$, $\mathsf{B}^4_1 = 1$, $\mathsf{C}^3_0 = 1/16$, $\mathsf{C}^3_1 = -1/2$, $\mathsf{C}^3_2 = 1$, $\mathsf{D}^4_0 = 1$, $\mathsf{D}^4_1 = -2$, $\mathsf{D}^4_2 = 1$ and so on.

In the following subsections we consider the rank one symmetric spaces $\mathbb{S}^n$, $\mathbb{R}\mathbf{P}^n$, $\mathbb{C}\mathbf{P}^n$, and $\mathbb{H}\mathbf{P}^n$ respectively, and give explicit formulations of the Maclaurin spectral coefficients for functions Φ of Laplace transform type, as in (3.1), acting on their respective Laplacians. We refer to Table 1 for the necessary spectral and geometric data, and to Table 2 for the respective values of a, b, and c associated to each space. Using these we calculate explicitly the Maclaurin heat coefficients $b_{2l}[e^{s\Delta}]$ for $l \geq 0$ and discuss and present some related examples.

3.1. **On the unit sphere $\mathbb{S}^n$.** In the following two theorems, for the odd and even dimensional case respectively, we will see how the Jacobi theta functions of the first and second kind naturally arise in the Maclaurin coefficients associated to functions $\Phi(-\Delta)$ of Laplace transform type.

Theorem 3.1 ($\mathbb{S}^n$, $n \geq 3$ odd)**.** *Take $\Phi(X)$ to be a function of Laplace transform type, as in (3.1). Then the Maclaurin spectral coefficients $b_{2l}[\Phi]$ for odd $n \geq 3$ can be written as*

$$b_0[\Phi] = \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{A}_m^n(-1)^{m+1}}{(n-1)!} \int_0^\infty f(s)\vartheta_1^{(m+1)}(s)\, d\mu, \tag{3.10}$$

where $d\mu = e^{s(n-1)^2/4}ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{\frac{n-3}{2}} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{A}_m^n(-1)^{j+m+1}\mathsf{c}_j^l}{(n-1)!} \binom{j}{i} \left(\frac{n-1}{2}\right)^{2i} \int_0^\infty f(s)\vartheta_1^{(m+j-i+1)}(s)\, d\mu, \tag{3.11}$$

where $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (2.7) specialised to the unit sphere with a,b,c as in Table 2, and the scalars A_m^n as in (3.6).

Proof. We begin by writing the multiplicty function M_k^n in a form that lets us apply (3.6). Indeed, writing $X_k = k + (n-1)/2$, we can refer to Table 1 to write the multiplicity as

$$M_k^n = (2k+n-1)\frac{\Gamma(k+n-1)}{k!(n-1)!} = \frac{2X_k}{(n-1)!}\prod_{j=1}^{n-2}(k+j). \tag{3.12}$$

We now note that each term $(k+j)$, for $j = 1, \dots, n-2$ of the product above can be written as $(X_k \pm j)$ for $j = 0, \dots, (n-3)/2$. Taking the product of $(X_k - j)$ and $(X_k + j)$ lets us apply (3.6) to write

$$M_k^n = \frac{2}{(n-1)!}\prod_{j=0}^{\frac{n-3}{2}}(X_k^2 - j^2) = \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{A}_m^n}{(n-1)!} X_k^{2m+2}. \tag{3.13}$$

Since the sum above vanishes when X_k is an integer between 1 and $(n-3)/2$, we can use the substitution $X_k \to p$ to write the heat trace $\operatorname{tr} e^{s\Delta}$ as

$$\begin{aligned}\operatorname{tr} e^{s\Delta} = \sum_{k=0}^\infty M_k^n e^{-s\lambda_k^n} &= \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{A}_m^n}{(n-1)!} \sum_{k=0}^\infty X_k^{2m+2} e^{-s\left(X_k^2-(n-1)^2/4\right)} \\ &= \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{A}_m^n e^{s(n-1)^2/4}}{(n-1)!} \sum_{p=1}^\infty p^{2m+2}e^{-sp^2} \\ &= \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{A}_m^n(-1)^{m+1}e^{s(n-1)^2/4}}{(n-1)!} \vartheta_1^{(m+1)}(s). \end{aligned}\tag{3.14}$$

Substituting the above into (3.2) and differentiating via Leibniz rule gives the results. □

In the even dimensional case, for $n \geq 2$, we have a similar formulation involving derivatives of the Jacobi theta function of the second kind.

Theorem 3.2 ($\mathbb{S}^n$, $n \geq 2$ even)**.** *Take $\Phi(X)$ as in (3.1). Then the Maclaurin spectral coefficients can be written as*

$$b_0[\Phi] = \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}_m^n(-1)^m}{(n-1)!} \int_0^\infty f(s)\vartheta_2^{(m)}(s)\, d\mu, \tag{3.15}$$

where $d\mu = e^{s(n-1)^2/4}ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{\frac{n-2}{2}} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{B}_m^n(-1)^{j+m}\mathsf{c}_j^l}{(n-1)!} \binom{j}{i} \left(\frac{n-1}{2}\right)^{2i} \int_0^\infty f(s)\vartheta_2^{(m+j-i)}(s)\, d\mu. \tag{3.16}$$

where $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (2.7) specialised to the unit sphere with a, b, c as in Table 2, and the scalars B_m^n as in (3.7).

Proof. This proof is similar to the proof of Theorem 3.1, so we skip some of the details. Taking $X_k = k+(n-1)/2$, we can express M_k^n in terms of the coefficients B_m^n from (3.7) (here we assume $n \geq 4$ as for $n = 2$ we can easily arrive at (3.17) below with $\mathsf{B}_0^2 = 1$ and without recourse to (3.7)) as

$$\begin{aligned} M_k^n = \frac{2X_k}{(n-1)!}\prod_{j=1}^{n-2}(k+j) &= \frac{2X_k}{(n-1)!}\prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}}(X_k^2 - j^2) \\ &= \sum_{m=0}^{\frac{n-2}{2}} \frac{2\mathsf{B}_m^n X_k}{(n-1)!} X_k^{2m}. \end{aligned} \tag{3.17}$$

Hence the heat trace $\operatorname{tr} e^{s\Delta}$ can be written as

$$\begin{aligned} \operatorname{tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k^n e^{-s\lambda_k^n} &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}_m^n e^{s(n-1)^2/4}}{(n-1)!} \sum_{p=\frac{1}{2}}^{\infty} 2p^{2m+1}e^{-sp^2} \\ &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}_m^n (-1)^m e^{s(n-1)^2/4}}{(n-1)!}\vartheta_2^{(m)}(s). \end{aligned} \tag{3.18}$$

The result follows when we substitute this formulation of the heat trace into (3.2) and differentiate. □

The polynomial $\mathscr{R}_l$ on the sphere, for $l = 0, 1, 2$. We now present explicit values of $b_{2l}[e^{s\Delta}] = \mathscr{R}_l(-d/ds)\operatorname{tr} e^{s\Delta}$ for $\mathbb{S}^n$, for $n = 2, 3, 4$. We have $\mathscr{R}_0(X) = 1$, $\mathscr{R}_1(X) = -X/n$, $\mathscr{R}_2(X) = (2-2n)/(n^2+2n)X + 3/(n^2+2n)X^2$. As a result we have:

- $\mathbb{S}^2$: $\operatorname{tr} e^{s\Delta} = \vartheta_2 e^{s/4}$, $\mathscr{R}_1(-d/ds)\operatorname{tr} e^{s\Delta} = [\vartheta_2'/2 + \vartheta_2/8]\, e^{s/4}$,
  $\mathscr{R}_2(-d/ds)\operatorname{tr} e^{s\Delta} = [3\vartheta_2'' + 7/2\vartheta_2' + 11/16\vartheta_2]\, e^{s/4}/8$.
- $\mathbb{S}^3$: $\operatorname{tr} e^{s\Delta} = -\vartheta_1' e^{s}/2$, $\mathscr{R}_1(-d/ds)\operatorname{tr} e^{s\Delta} = -[\vartheta_1'' + \vartheta_1']\, e^{s}/6$,
  $\mathscr{R}_2(-d/ds)\operatorname{tr} e^{s\Delta} = -[\vartheta_1'''/10 + \vartheta_1''/3 + 7/30\vartheta_1']\, e^{s}$.
- $\mathbb{S}^4$: $\operatorname{tr} e^{s\Delta} = -[\vartheta_2'/6 + \vartheta_2/24]\, e^{9s/4}$,
  $\mathscr{R}_1(-d/ds)\operatorname{tr} e^{s\Delta} = -[\vartheta_2'' + 5/2\vartheta_2' + 9/16\vartheta_2]\, e^{9s/4}/24$,
  $\mathscr{R}_2(-d/ds)\operatorname{tr} e^{s\Delta} = -[\vartheta_2'''/3 + 9/4\vartheta_2'' + 179/48\vartheta_2' + 51/64\vartheta_2]\, e^{9s/4}/16$.

3.2. **On the real projective space $\mathbb{RP}^n$.** Before stating the results below we introduce half series of the odd and even terms that make up the first and second theta functions. Taking $\vartheta_{1,o}(s)$ as the sum of the odd terms of ϑ_1, and $\vartheta_{1,e}(s)$ as the sum of the even terms, we can write $\vartheta_{1o}(s) + \vartheta_{1,e}(s) = \vartheta_1(s)$. We define these explicitly as

$$\vartheta_{1,o}(s) = \sum_{j\in\mathbb{Z}} e^{-s(2j+1)^2}, \quad \vartheta_{1,e}(s) = \sum_{j\in\mathbb{Z}} e^{-s(2j)^2}. \tag{3.19}$$

Note that by a basic inspection we have $\vartheta_{1,e}(s) = \vartheta_1(4s)$, $\vartheta_{1,o}(s) = \vartheta_1(s) - \vartheta_1(4s)$. Likewise for ϑ_2, we define $\vartheta_{2,o}(s)$ and $\vartheta_{2,e}(s)$ so that $\vartheta_{2,o}(s) + \vartheta_{2,e}(s) = \vartheta_2(s)$ as

$$\vartheta_{2,o}(s) = \sum_{j=0}^{\infty} (4j+1) e^{-s(2j+1/2)^2}, \quad \vartheta_{2,e}(s) = \sum_{j=0}^{\infty} (4j+3) e^{-s(2j+3/2)^2}. \tag{3.20}$$

Theorem 3.3 ($\mathbb{RP}^n$, $n \geq 3$ odd)**.** *Take $\Phi(X)$ as in (3.1). Then the Maclaurin spectral coefficients $b_{2l}[\Phi]$ can be written as*

$$b_0[\Phi] = \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{A}_m^n (-1)^{m+1}}{(n-1)!} \int_0^\infty f(s) \vartheta_{1,\star}^{(m+1)}(s)\, d\mu, \tag{3.21}$$

where $d\mu = e^{s(n-1)^2/4} ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{\frac{n-3}{2}} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{A}_m^n (-1)^{j+m+1} \mathsf{c}_j^l}{(n-1)!} \binom{j}{i} \left(\frac{n-1}{2}\right)^{2i} \int_0^\infty f(s) \vartheta_{1,\star}^{(m+1+j-i)}(s)\, d\mu, \tag{3.22}$$

where we take $\vartheta_{1,o}$ when $(n-1)/2$ is odd, and $\vartheta_{1,e}$ when $(n-1)/2$ is even. Here $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (2.7) specialised to the real projective space with a, b, c as in Table 2, and the scalars A_m^n as in (3.6).

Proof. Taking $X_k = 2k + (n-1)/2$, we can use (3.6) to express M_k^n in terms of the coefficients A_m^n,

$$M_k^n = \frac{2X_k}{(n-1)!} \prod_{j=1}^{n-2} (2k+j) = \frac{2}{(n-1)!} \prod_{j=0}^{\frac{n-3}{2}} \left(X_k^2 - j^2\right) = \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{A}_m^n}{(n-1)!} X_k^{2m+2}. \tag{3.23}$$

This lets us write the heat trace as

$$\operatorname{tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k^n e^{-s\lambda_k^n} = \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{A}_m^n e^{s(n-1)^2/4}}{(n-1)!} \sum_{k=0}^{\infty} X_k^{2m+2} e^{-sX_k^2}$$
$$= \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{A}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \frac{d^{m+1}}{ds^{m+1}} \sum_{k=0}^{\infty} e^{-sX_k^2}. \tag{3.24}$$

Here we note that X_k takes odd integer values when $(n-1)/2$ is odd, and even integer values when $(n-1)/2$ is even. With this in mind we can consider both of these cases separately, and referring to (3.19) we have the following formulations.

(i) When $(n-1)/2$ is odd, we substitute $X_k \to 2j+1$ and extend the sums so they run over $\mathbb{Z}$ by noting that (3.23) gives that M_k^n vanishes when X_k is a integer between 0 and $(n-3)/2$.

$$\operatorname{tr} e^{s\Delta} = \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{A}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \frac{d^{m+1}}{ds^{m+1}} \sum_{j=\frac{n-3}{4}}^{\infty} e^{-s(2j+1)^2}$$
$$= \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{A}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \frac{d^{m+1}}{ds^{m+1}} \sum_{j\in\mathbb{Z}} e^{-s(2j+1)^2}$$
$$= \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{A}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \vartheta_{1,o}^{(m+1)}(s). \tag{3.25}$$

(ii) When $(n-1)/2$ is even, we use the substitution $X_k = 2j$ and extend the sums as above to write

$$\operatorname{tr} e^{s\Delta} = \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{A}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \frac{d^{m+1}}{ds^{m+1}} \sum_{j=\frac{n-1}{4}}^{\infty} e^{-s(2j)^2}$$
$$= \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{A}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \vartheta_{1,e}^{(m+1)}(s). \tag{3.26}$$

The result follows when we substitute these formulations of the heat trace into (3.2) and differentiate appropriately. □

Theorem 3.4 ($\mathbb{R}\mathbf{P}^n$, $n \geq 2$ even)**.** *Take* $\Phi(X)$ *as in* (3.1)*. Then the Maclaurin spectral coefficients can be written as*

$$b_0[\Phi] = \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}_m^n}{(n-1)!} \int_0^{\infty} f(s)\vartheta_{2,\star}^{(m)}(s)\, d\mu, \tag{3.27}$$

where $d\mu = e^{s(n-1)^2/4}ds$, *and for* $l > 0$ *we have*

$$b_{2l}[\Phi] = \sum_{m=0}^{\frac{n-2}{2}} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{B}^n_m (-1)^{j+m} \mathsf{c}^l_j}{(n-1)!} \binom{j}{i} \left(\frac{n-1}{2}\right)^{2i} \int_0^\infty f(s) \vartheta^{(m+j-i)}_{2,\star}(s)\, d\mu. \tag{3.28}$$

where we take $\vartheta_{2,o}$ *for* $n/2$ *odd, and* $\vartheta_{2,e}$ *for* $n/2$ *even. Here* $\mathsf{c}^l_j = \mathsf{c}^l_j(a,b,c)$ *are the hypergeometric coefficients as in* (2.7) *specialised to the real projective space with* a, b, c *as in Table* 2*, and the scalars* B^n_m *as in* (3.7)*.*

Proof. As before we begin by writing the multiplicity function in terms of a polynomial. We set $X_k = 2k + (n-1)/2$ and for $n \geq 4$ write

$$\begin{aligned} M^n_k = \frac{2X_k}{(n-1)!} \prod_{j=1}^{n-2} (2k+j) &= \frac{2X_k}{(n-1)!} \prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}} \left(X_k^2 - j^2\right) \\ &= \frac{2X_k}{(n-1)!} \prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}} \left(X_k^2 - j^2\right) \\ &= \sum_{m=0}^{\frac{n-2}{2}} \frac{2\mathsf{B}^n_m}{(n-1)!} X_k^{2m+1}, \end{aligned} \tag{3.29}$$

(note that the last equation remains true for $n = 2$). Next substituting for M^n_k

$$\begin{aligned} \operatorname{tr} e^{s\Delta} = \sum_{k=0}^{\infty} M^n_k e^{-s\lambda^n_k} &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}^n_m e^{s(n-1)^2/4}}{(n-1)!} \sum_{k=0}^{\infty} 2X_k (X_k)^{2m} e^{-s(X_k)^2} \\ &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}^n_m e^{s(n-1)^2/4}}{(n-1)!} \sum_{k=0}^{\infty} \frac{d^m}{ds^m} 2X_k e^{-s(X_k)^2}. \end{aligned} \tag{3.30}$$

Here the sequence $2X_k$ for $k = 0, 1, 2, \dots$ takes values $4j+1$ for $j = 0, 1, \dots$ when $n/2$ is odd, and takes values $4j+1$ when $n/2$ is even.

(i) When $n/2$ is odd, we substitute $X_k \to 2j + 1/2$. We then extend the sums to $j = 0$ by noting the sum in (3.29) vanishes for $X_k = 1/2, 3/2, \dots, (n-2)/2 - 1/2$.

$$\begin{aligned} \operatorname{tr} e^{s\Delta} &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}^n_m e^{s(n-1)^2/4}}{(n-1)!} \sum_{j=\frac{n-2}{4}}^{\infty} \frac{d^m}{ds^m} (4j+1) e^{-s(2j+1/2)^2} \\ &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}^n_m e^{s(n-1)^2/4}}{(n-1)!} \sum_{j=0}^{\infty} \frac{d^m}{ds^m} (4j+1) e^{-s(2j+1/2)^2} \\ &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}^n_m e^{s(n-1)^2/4}}{(n-1)!} \vartheta^{(m)}_{2,o}(s). \end{aligned} \tag{3.31}$$

(ii) When $(n-1)/2$ is even, we use the substitution $X_k = 2j + 3/2$ and extend the sums as above to write

$$\begin{aligned}
\operatorname{tr} e^{s\Delta} &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}_m^n e^{s(n-1)^2/4}}{(n-1)!} \sum_{j=\frac{n-4}{4}}^{\infty} \frac{d^m}{ds^m}(4j+3)e^{-s(2j+3/2)^2} \\
&= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}_m^n e^{s(n-1)^2/4}}{(n-1)!} \sum_{j=0}^{\infty} \frac{d^m}{ds^m}(4j+3)e^{-s(2j+3/2)^2} \\
&= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}_m^n e^{s(n-1)^2/4}}{(n-1)!} \vartheta_{2,e}^{(m)}(s). \qquad (3.32)
\end{aligned}$$

The heat trace formula can then be written as

$$\operatorname{tr} e^{s\Delta} = \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{B}_m^n e^{s(n-1)^2/4}}{(n-1)!} \vartheta_{2,\star}^{(m)}(s), \qquad (3.33)$$

where we take $\vartheta_{2,o}$ when $n/2$ is odd, and $\vartheta_{2,e}$ when $n/2$ is even. Substituting (3.33) into (3.2) and differentiating gives the result. ☐

3.3. **On the complex projective space $\mathbb{C}\mathbf{P}^n$.** On $\mathbb{C}\mathbf{P}^n$ for n odd, we see that the theta function ϑ_2 naturally arises in the Maclaurin coefficients, and for n even we similarly see ϑ_3. This is in contrast to the earlier cases of $\mathbb{S}^n$ and $\mathbb{R}\mathbf{P}^n$, where we saw ϑ_1 and ϑ_2 instead.

Theorem 3.5 ($\mathbb{C}\mathbf{P}^n$, $n \geq 3$ odd)**.** *Let the function $\Phi(X)$ be of Laplace transform type as in (3.1). Then the Maclaurin coefficients $b_{2l}[\Phi]$ in odd dimensions $n \geq 3$ are given explicitly by*

$$b_0[\Phi] = \sum_{m=0}^{n-1} \frac{\mathsf{C}_m^n (-1)^m}{n!(n-1)!} \int_0^\infty \vartheta_2^{(m)}(s) f(s)\, d\mu, \qquad (3.34)$$

where $d\mu = d\mu(s) = e^{sn^2/4} ds$, and for $l > 0$

$$b_{2l}[\Phi] = \sum_{m=0}^{n-1} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{C}_m^n (-1)^{j+m} \mathsf{c}_j^l}{n!(n-1)!} \binom{j}{i} \left(\frac{n}{2}\right)^{2i} \int_0^\infty f(s) \vartheta_2^{(m+j-i)}(s)\, d\mu, \qquad (3.35)$$

where the coefficients $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (2.7) specialised to the complex projective space with a, b, c as in Table 2, and C_m^n are the scalars defined in (3.8).

Proof. Defining $X_k = k + n/2$, we can write the multiplicity function M_k^n of the eigenvalues of the Laplacian on $\mathbb{C}\mathbf{P}^n$ as

$$M_k^n = \frac{2k+n}{n} \left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2 = \frac{2X_k}{n!(n-1)!} \prod_{j=1}^{n-1} (k+j)^2. \qquad (3.36)$$

We can write the product above in terms of X_k by noting pairs of terms of the form $(k+j)$ for $j = 1, 2, \ldots, n-1$ can be multiplied together to get terms of the form $(X_k^2 - j^2)$ for $j = 1/2, 3/2, \ldots...(n-2)/2$. Applying (3.8), this leaves us with

$$M_k^n = \frac{2X_k}{n!(n-1)!} \prod_{j=\frac{1}{2}}^{\frac{n-3}{2}+\frac{1}{2}} (X_k^2 - j^2)^2 = \frac{2X_k}{n!(n-1)!} \sum_{m=0}^{n-1} \mathsf{C}_m^n X_k^{2m}, \tag{3.37}$$

Hence we can write the heat trace $\mathrm{tr}\, e^{s\Delta}$ as

$$\begin{aligned} \mathrm{tr}\, e^{s\Delta} = \sum_{k=0}^{\infty} M_k^n e^{-s\lambda_k^n} &= \sum_{m=0}^{n-1} \frac{2\mathsf{C}_m^n}{n!(n-1)!} \sum_{k=0}^{\infty} X_k^{2m+1} e^{-s(X_k^2 - n^2/4)} \\ &= \sum_{m=0}^{n-1} \frac{\mathsf{C}_m^n (-1)^m e^{sn^2/4}}{n!(n-1)!} \sum_{j=0}^{\infty} (2j+1) \frac{d^m}{ds^m} \left[e^{-s(j+1/2)^2} \right] \\ &= \sum_{m=0}^{n-1} \frac{\mathsf{C}_m^n (-1)^m e^{sn^2/4}}{n!(n-1)!} \vartheta_2^{(m)}(s). \end{aligned} \tag{3.38}$$

Here we have noted that (3.37) implies that the multiplicity vanishes when X_k is $1/2, 3/2, \ldots, (n-3)/2 + 1/2$. This lets us extend the sum in the third step above. Differentiating via Leibniz rule and substituting the result into (3.1) gives the solution. □

Theorem 3.6 ($\mathbb{C}\mathbf{P}^n$, $n \geq 2$ even)**.** *Take $\Phi(X)$ as in (3.1). Then for even $n \geq 2$, the Maclaurin coefficients $b_{2l}[\Phi]$ can be expressed as*

$$b_0[\Phi] = \sum_{m=0}^{n-2} \frac{(-1)^{m+1} \mathsf{D}_m^n}{n!(n-1)!} \int_0^{\infty} \vartheta_3^{(m+1)}(s) f(s)\, d\mu, \tag{3.39}$$

where $d\mu = d\mu(s) = e^{sn^2/4} ds$ and for $l > 0$,

$$b_{2l}[\Phi] = \sum_{m=0}^{n-2} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{D}_m^n (-1)^{j+m+1} \mathsf{c}_j^l}{n!(n-1)!} \binom{j}{i} \left(\frac{n}{2}\right)^{2i} \int_0^{\infty} f(s) \vartheta_3^{(m+j-i+1)}(s)\, d\mu, \tag{3.40}$$

where the coefficients $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (2.7) specialised to the complex projective space with a, b, c as in Table 2, and D_m^n are the scalars defined in (3.9).

Proof. Taking $X_k = k + n/2$ as in the proof of the previous theorem, we can then write the multiplicity as

$$\begin{aligned} M_k^n = \frac{2k+n}{n!(n-1)!} \left[\frac{\Gamma(k+n)}{k!} \right]^2 &= \frac{2X_k}{n!(n-1)!} \prod_{j=1}^{n-1} (k+j)^2 \\ &= \frac{2X_k^3}{n!(n-1)!} \prod_{j=1}^{\frac{n-2}{2}} (X_k^2 - j^2)^2. \end{aligned} \tag{3.41}$$

In the last equality we have noted that each $(k+j)$ from the product over $j = 0, \ldots, n-1$ can be written as $(X_k \pm j)$ for $j = 1, \ldots, (n-2)/2$. Factoring out the

$j = 0$ terms and taking the product of $(X_k - j)$ with $(X_k + j)$ gives the required result. Now applying (3.9) gives

$$M_k^n = \frac{2X_k^3}{n!(n-1)!} \sum_{m=0}^{n-2} \mathsf{D}_m^n X_k^{2m}, \tag{3.42}$$

Inserting this formulation into the heat trace, and using that the above sum vanishes for X_k an integer between 1 and $(n-2)/2$, we can apply the substitution $X_k \to p$ to write

$$\begin{aligned} \operatorname{tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k^n e^{-s\lambda_k^n} &= \sum_{m=0}^{n-2} \frac{2\mathsf{D}_m^n}{n!(n-1)!} \sum_{p=1}^{\infty} p^{2m+3} e^{-s(p^2-n^2/4)} \\ &= \sum_{m=0}^{n-2} \frac{\mathsf{D}_m^n e^{s(n/4)^2}}{n!(n-1)!} (-1)^{m+1} \vartheta_3^{(m+1)}(s). \end{aligned} \tag{3.43}$$

The result follows after differentiating this via Leibniz rule. □

The polynomial $\mathscr{R}_l$ on the $\mathbb{C}\mathbf{P}^n$, for $l = 0, 1, 2$. We now present explicit values of $b_{2l}[e^{s\Delta}] = \mathscr{R}_l(-d/ds)\operatorname{tr} e^{s\Delta}$ on $\mathbb{C}\mathbf{P}^n$, for $n = 1, 2, 3, 4$. We have $\mathscr{R}_0(X) = 1$, $\mathscr{R}_1(X) = -X/2n$, $\mathscr{R}_2(X) = -X/4n + 3/(4n(n+1))X^2$. As a result we have:

- $\mathbb{C}\mathbf{P}^1$: $\Theta(s) = \operatorname{tr} e^{s\Delta} = \vartheta_2 e^{s/4}$, $\mathscr{R}_1(-d/ds)\Theta(s) = [\vartheta_2'/2 + \vartheta_2/8]\, e^{s/4}$, $\mathscr{R}_2(-d/ds)\Theta(s) = [3\vartheta_2'' + 7/2\vartheta_2' + 11/16\vartheta_2]\, e^{s/4}/8$.
- $\mathbb{C}\mathbf{P}^2$: $\Theta(s) = \operatorname{tr} e^{s\Delta} = -\vartheta_3' e^{s/4}/2$, $\mathscr{R}_1(-d/ds)\Theta(s) = -[\vartheta_3'' + \vartheta_3']\, e^{s}/8$, $\mathscr{R}_2(-d/ds)\Theta(s) = -[\vartheta_3''' + 3\vartheta_3'' + 2\vartheta_3']\, e^{s}/16$.
- $\mathbb{C}\mathbf{P}^3$: $\Theta(s) = \operatorname{tr} e^{s\Delta} = [\vartheta_2''/4 + \vartheta_2'/8 + \vartheta_2/64]\, e^{9s/4}/3$, $\mathscr{R}_1(-d/ds)\Theta(s) = [\vartheta_2''' + 11/4\vartheta_2'' + 19/16\vartheta_2' + 9/64\vartheta_2]\, e^{9s/4}/72$, $\mathscr{R}_2(-d/ds)\Theta = [\vartheta_2'''' + 19/3\vartheta_2''' + 265/24\vartheta_2'' + 211/48\vartheta_2' + 129/256\vartheta_2] e^{9s/4}/192$.
- $\mathbb{C}\mathbf{P}^4$: $\Theta(s) = \operatorname{tr} e^{s\Delta} = -[\vartheta_3''' + 2\vartheta_3'' + \vartheta_3']\, e^{4s}/144$, $\mathscr{R}_1(-d/ds)\Theta(s) = -[\vartheta_3''''/9 + 2/3\vartheta_3''' + \vartheta_3'' + 4/9\vartheta_3']\, e^{4s}/128$, $\mathscr{R}_2(-d/ds)\Theta(s) = -\left[3\vartheta_3^{(5)} + 35\vartheta_3'''' + 129\vartheta_3''' + 165\vartheta_3'' + 68\vartheta_3'\right] e^{4s}/11520$.

3.4. **On the quaternionic projective space $\mathbb{H}\mathbf{P}^n$.** Before the statement of the theorem below we require the introduction of some notation. We define the scalars E_m^n as the coefficients of X^{2m} in the polynomial below.

$$\prod_{j=\frac{1}{2}}^{n-\frac{1}{2}} (X^2 - j^2) \prod_{j=\frac{1}{2}}^{n-\frac{3}{2}} (X^2 - j^2) = \sum_{m=0}^{2n-1} \mathsf{E}_m^n X^{2m}. \tag{3.44}$$

Note the clear relation to the scalars B_m^n defined in (3.7). Indeed, we can write

$$\sum_{m=0}^{2n-1} \mathsf{E}_m^n X^{2m} = \sum_{m=0}^{n} \mathsf{B}_m^{2n+2} X_k^{2m} \sum_{m=0}^{n-1} \mathsf{B}_m^{2n} X_k^{2m}. \tag{3.45}$$

For the first few coefficients we have $\mathsf{E}_0^2 = -9/64$, $\mathsf{E}_1^2 = 19/16$, $\mathsf{E}_2^2 = -11/4$, $\mathsf{E}_3^2 = 1$. Below we also set $\mathsf{E}_0^1 = -1/4$, $\mathsf{E}_1^1 = 1$.

Theorem 3.7 ($n \geq 1$, $\mathbb{H}\mathbf{P}^n$)**.** *Take $\Phi(X)$ to be a function of Laplace transform type as in (3.1). Then the Maclaurin spectral coefficients $b_{2l}[\Phi]$ are given by*

$$b_0[\Phi] = \sum_{m=0}^{2n-1} \frac{(-1)^m \mathsf{E}_m^n}{(2n-1)!(2n+1)!} \int_0^\infty f(s) \vartheta_2^{(m)}(s)\, d\mu, \tag{3.46}$$

where $d\mu = d\mu(s) = e^{s(2n+1)^2/4} ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{2n-1} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{(-1)^{m+j} \mathsf{E}_m^n \mathsf{c}_j^l}{(2n-1)!(2n+1)!} \binom{j}{i} \left(\frac{2n+1}{2}\right)^{2i} \int_0^\infty f(s) \vartheta_2^{(m+j-i)}(s)\, d\mu \tag{3.47}$$

where the coefficients $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (2.7) specialised to the quaternionic projective space with a, b, c as in Table 2, and E_m^n are the scalars defined in (3.44).

Proof. Setting $X_k = k + n + 1/2$, we can write the multiplicity function M_k^n as

$$\begin{aligned} M_k^n &= \frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)} \left(\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right)^2 \\ &= \frac{2X_k(k+2n)}{(2n-1)!(2n+1)!(k+1)} \prod_{j=1}^{2n-1} (k+j)^2 \\ &= \frac{2X_k}{(2n-1)!(2n+1)!} \prod_{j=\frac{1}{2}}^{n-\frac{1}{2}} (X_k^2 - j^2) \prod_{j=\frac{1}{2}}^{n-\frac{3}{2}} (X_k^2 - j^2) \\ &= \frac{2X_k}{(2n-1)!(2n+1)!} \sum_{m=0}^{2n-1} \mathsf{E}_m^n X^{2m} \end{aligned} \tag{3.48}$$

Hence we can write the heat trace as

$$\begin{aligned} \operatorname{tr} e^{s\Delta} = \sum_{k=0}^\infty M_k^n e^{-s\lambda_k^n} &= \sum_{m=0}^{2n-1} \frac{2\mathsf{E}_m^n e^{s(2n-1)^2/4}}{(2n-1)!(2n+1)!} \sum_{k=0}^\infty X_k^{2m+1} e^{-sX_k^2} \\ &= \sum_{m=0}^{2n-1} \frac{(-1)^m \mathsf{E}_m^n e^{s(2n-1)^2/4}}{(2n-1)!(2n+1)!} \frac{d^m}{ds^m} \sum_{k=0}^\infty 2X_k e^{-sX_k^2}. \end{aligned} \tag{3.49}$$

Here we can use the substitution $X_k \to p$, and the extend the sum to zero by noting that (3.48) shows that the multiplicity vanishes when $X_k - 1/2$ is an integer between 0 and $n-1$.

$$\begin{aligned} \operatorname{tr} e^{s\Delta} &= \sum_{m=0}^{2n-1} \frac{(-1)^m \mathsf{E}_m^n e^{s(2n+1)^2/4}}{(2n-1)!(2n+1)!} \frac{d^m}{ds^m} \sum_{p=n+\frac{1}{2}}^\infty 2pe^{-sp^2} \\ &= \sum_{m=0}^{2n-1} \frac{(-1)^m \mathsf{E}_m^n e^{s(2n+1)^2/4}}{(2n-1)!(2n+1)!} \vartheta_2^{(m)}(s). \end{aligned} \tag{3.50}$$

Substituting this formulation of the trace into (3.2) and differentiating via Leibniz rule gives the result. □

The polynomial $\mathscr{R}_l$ on the $\mathbb{H}\mathbf{P}^n$, for $l = 0, 1, 2$. We now present explicit values of $\mathscr{R}_l(-d/ds)\mathrm{tr}\, e^{s\Delta}$ on $\mathbb{H}\mathbf{P}^n$, for $n = 1, 2$. We have $\mathscr{R}_0(X) = 1$, $\mathscr{R}_1(X) = -X/4n$, $\mathscr{R}_2(X) = -(n+2)/(8n^2 - 4n)X + 3/(16^2 + 8n)X^2$. As a result we can write:

- $\mathbb{H}\mathbf{P}^1$: $\Theta(s) = \mathrm{tr}\, e^{s\Delta} = -\left[\vartheta_2' + \vartheta_2/4\right] e^{9s/4}/6$,
  $\mathscr{R}_1(-d/ds)\Theta(s) = -\left[\vartheta_2'' + 5/2\vartheta_2' + 9/16\vartheta_2\right] e^{9s/4}/24$,
  $\mathscr{R}_2(-d/ds)\Theta(s) = -\left[\vartheta_2'''/3 + 9/4\vartheta_2'' + 179/48\vartheta_2' + 51/64\vartheta_2\right] e^{9s/4}/16$.
- $\mathbb{H}\mathbf{P}^2$: $\Theta(s) = \mathrm{tr}\, e^{s\Delta} = -\left[\vartheta_2''' + 11/4\vartheta_2'' + 19/16\vartheta_2' + 9/64\vartheta_2\right] e^{25s/4}/720$,
  $\mathscr{R}_1(-d/ds)\Theta(s) = -[16\vartheta_2'''' + 144\vartheta_2''' + 294\vartheta_2'' + 121\vartheta_2' + 225/16\vartheta_2]e^{25s/4}/92160$,
  $\mathscr{R}_2(-d/ds)\Theta(s) = -[3072\vartheta_2^{(5)} + 55040\vartheta_2'''' + 302976\vartheta_2''' + 526560\vartheta_2'' + 209852\vartheta_2' + 24075\vartheta_2]e^{25s/4}/58982400$.

Remark 3.8. As a result of the well-known identifications $\mathbb{H}\mathbf{P}^1 \cong \mathbb{S}^4$ and $\mathbb{C}\mathbf{P}^1 \cong \mathbb{S}^2$ we have all the corresponding quantities calculated above agreeing in these special cases respectively.

4. Further extension to generalised hypergeometric functions

The hypergeometric function ${}_2F_1 = F(a, b; c; z)$ encountered earlier in the paper is a special case of the generalised hypergeometric function ${}_pF_q(z) = {}_pF_q(\mathbf{a}; \mathbf{b}; z)$, where $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{b} = (b_1, \dots, b_q)$ with $a_1, \dots a_p, b_1, \dots, b_q \in \mathbb{C}$ and none of the parameters $b_1, \dots, b_q$ being non-positive integers (see [3, 16, 17]). As a matter of fact here the generalised hypergeometric function is formally defined by the power series

$$ {}_pF_q(\mathbf{a}; \mathbf{b}; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{z^k}{k!}. \tag{4.1} $$

If any of $a_1, \dots, a_p$ is a non-positive integer then the series (4.1) terminates and represents a polynomial in z (of degree $-a_l$ where a_l is the largest of the non positive integers in the list) and so the series converges for all z. For the remaining cases we have the following convergence criteria, depending on the relative values of p and q.

- If $p > q + 1$ the series diverges everywhere except $z = 0$.
- If $p < q + 1$ the series converges for all z and so ${}_pF_q$ is an entire function.
- If $p = q + 1$ (as with ${}_2F_1(z)$) the series converges for $|z| < 1$ and diverges for $|z| > 1$. The series may converge or diverge at points on the unit circle and depending on the parameters $\mathbf{a}$ and $\mathbf{b}$ various convergence criteria are available. Note that in this case the series admits an analytic continuation along any path avoiding its branch point $z = 1$ to the exterior of the unit disk.

Defining the operator $\vartheta = zd/dz$, the generalised hypergeometric function $w = {}_pF_q(\mathbf{a}; \mathbf{b}; z)$ satisfies a differential equation of order $\max(p, q + 1)$, given by,

$$ \left(\vartheta \left(\vartheta + b_1 - 1\right) \dots \left(\vartheta + b_q - 1\right) - z \left(\vartheta + a_1\right) \dots \left(\vartheta + a_p\right)\right) w = 0. \tag{4.2} $$

It is interesting however to point out that for certain values of p and q one can give explicit examples of differential equations of lower order than expected that can be satisfied by the generalised hypergeometric function (*see*, e.g., [22]).

It is straightforward to derive generalisations of the differential identities stated in Appendix A for ${}_2F_1(a,b;c;z)$ in this context. Indeed, here, the counterpart of (A.4) takes the form

$$\begin{aligned}\frac{d^m}{dz^m}{}_pF_q(\mathbf{a};\mathbf{b};z) &= \frac{\prod_{i=1}^p(a_i)_m}{\prod_{j=1}^q(b_j)_m}\sum_{k=0}^\infty \frac{(a_1+m)_k(a_2+m)_k\dots(a_p+m)_k}{(b_1+m)_k(b_2+m)_k\dots(b_q+m)_k}\frac{z^k}{k!}\\ &= \frac{\prod_{i=1}^p(a_i)_m}{\prod_{j=1}^q(b_j)_m}{}_pF_q(\mathbf{a}+m;\mathbf{b}+m;z),\end{aligned} \tag{4.3}$$

where $\mathbf{a}+m$ denotes $(a_1+m,\dots,a_p+m)$ and $\mathbf{b}+m$ denotes $(b_1+m,\dots,b_p+m)$. The statement and formulation of Lemma 2.1 now naturally involves a product of p Pochhammer symbols, and as such can be written as

$$(a_1)_j(a_2)_j\dots(a_p)_j = \prod_{k=0}^{j-1}\prod_{i=1}^p(a_i+k) = \sum_{l=1}^j \mathsf{d}_{l,j}(\mathbf{a})\left[\prod_{i=1}^p a_i\right]^l. \tag{4.4}$$

In this case the scalars $\mathsf{d}_{l,j}(\mathbf{a})$ are the coefficients of the 'eigenvalue' $\prod_{i=1}^p a_i$ in the above polynomial. With the operator $\mathscr{L}_{\mathsf{P}} = \mathsf{P}_d(d/d\theta)$ as in (1.5), we can then state the following theorem.

Theorem 4.1. *Let $\mathscr{L}_{\mathsf{P}}$ be the differential operator as defined in (1.5). Then for $\mathbf{a}=(a_1,\dots,a_p)$ and $\mathbf{b}=(b_1,\dots,b_q)$, with each b_j not a non-positive integer, the generalised hypergeometric function ${}_pF_q(\mathbf{a};\mathbf{b};z)$ satisfies the differential identity*

$$\mathscr{L}_{\mathsf{P}}\left[{}_pF_q\right]\left(\mathbf{a};\mathbf{b};\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor d/2\rfloor} p_{2m}\sum_{j=1}^m \mathsf{c}_j^m(\mathbf{a},\mathbf{b})\left[-\prod_{i=1}^p a_i\right]^j, \tag{4.5}$$

where the scalars $\mathsf{c}_j^m(\mathbf{a},\mathbf{b})$, called the generalised hypergeometric coefficients, are explicitly given by the formula (with $1\le j\le m$)

$$\mathsf{c}_j^m(\mathbf{a},\mathbf{b}) = (-1)^j\sum_{i=j}^m \frac{\mathsf{b}_i^m\mathsf{d}_{j,i}}{(-2)^i\prod_{k=0}^{i-1}\prod_{l=1}^q(b_l+k)}. \tag{4.6}$$

Furthermore the set of scalars b_i^m and $\mathsf{d}_{j,i}=\mathsf{d}_{j,i}(\mathbf{a})$ are defined by (B.6) and (4.4) respectively.

Appendix A. The hypergeometric function ${}_2F_1(z) = F(a,b;c;z)$

The hypergeometric function $F={}_2F_1(z)$ is defined in the unit disk – its circle of convergence – by the power series

$$F(a,b;c;z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{A.1}$$

with the parameters $a, b, c \in \mathbb{C}$ and c not a non-positive integer, that is, $c \neq 0, -1, -2, \cdots$. Here $(a)_m$ denotes the Pochhammer symbol (or the rising factorial) defined by $(a)_0 = 1$ and for $m \geq 1$ by

$$(a)_m = a(a+1)\dots(a+m-1) = \frac{\Gamma(a+m)}{\Gamma(a)}, \tag{A.2}$$

where the second equality assumes that a is not a non-positive integer. Note that from (A.1) we have the point-wise identity $F(a, b; c; 0) = 1$ that was used earlier in Section 2. The hypergeometric function satisfies the differential identity

$$\frac{d}{dz}F(a, b; c; z) = \frac{ab}{c}F(a+1, b+1; c+1; z), \tag{A.3}$$

from which one obtains upon a straightforward iteration and for $m \geq 1$:

$$\frac{d^m}{dz^m}F(a, b; c; z) = \frac{(a)_m(b)_m}{(c)_m}F(a+m, b+m; c+m; z). \tag{A.4}$$

The hypergeometric function is a solution to the hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2} + (c - (a+b+1)\,z)\,\frac{dw}{dz} - abw = 0. \tag{A.5}$$

This equation can be obtained by a suitable change of variables from any second-order ordinary differential equation with at most three regular singular points. The change of variables $z = (1-t)/2$ and the choice of parameters $a = -k, b = \alpha+\beta+k+1$ and $c = \alpha+1$ transforms (A.5) into the well-known Jacobi differential equation,

$$(1-t^2)\frac{d^2w}{dt^2} + (\beta - \alpha - (\alpha+\beta+2)t)\frac{dw}{dt} + k(\alpha+\beta+k+1)w = 0. \tag{A.6}$$

The Jacobi polynomial $w = \mathscr{P}_k^{(\alpha,\beta)}(t)$ encountered before constitutes a solution to this equation and as such is a special case of the hypergeometric function.

Appendix B. The Bell polynomial $\mathsf{B}_{m,j}$ and Faà di Bruno's formula

To prove the main theorem of Section 2 we make use of Faà di Bruno's formula, a generalised chain rule, in order to write derivatives of $F(a, b; c; z)$ in terms of (incomplete) Bell polynomials $\mathsf{B}_{m,j}(x)$ (see [2] pp. 204-207 and [9]). These are defined for $x = (x_1, \dots, x_{m-j+1})$ as

$$\mathsf{B}_{m,j}(x) = \sum \frac{m!}{k_1!k_2!\dots k_{m-j+1}!} \prod_{i=1}^{m-j+1} \left(\frac{x_i}{i!}\right)^{k_i} \tag{B.1}$$

where the sum is taken over all sequences of non-negative integers $k_1, \dots, k_{m-j+1}$ such that

$$k_1 + \cdots + k_{m-j+1} = j, \text{ and } k_1 + 2k_2 + \cdots + (m-j+1)k_{m-j-1} = m. \tag{B.2}$$

For smooth functions f, g, Faà di Bruno's formula then asserts that

$$\frac{d^m}{dx^m}f(g(x)) = \sum_{j=1}^{m} f^{(j)}(g(x)) \cdot \mathsf{B}_{m,j}\left(g'(x), g''(x), \dots, g^{(m-j+1)}(x)\right). \tag{B.3}$$

Setting $g(x) = \cos x$ in Faà di Bruno's formula and evaluating at $x = 0$ results in the special case

$$\frac{d^{2m}}{dx^{2m}} f(\cos x)\bigg|_{x=0} = \left\{ \sum_{j=1}^{m} \mathsf{b}_j^m \frac{d^j}{dt^j} f(t) \right\}\bigg|_{t=1}, \tag{B.4}$$

where the coefficients

$$\mathsf{b}_j^m = \mathsf{B}_{2m,j}\left(-\sin x, -\cos x, \sin x, \dots\right)\Big|_{x=0} \tag{B.5}$$

satisfy the recursive formula, for $m, j > 0$ integers

$$\mathsf{b}_j^m = \begin{cases} (-1)^m & \text{if } j = 1 \\ -\left(j^2 \mathsf{b}_j^{m-1} + (2j-1)\mathsf{b}_{j-1}^{m-1}\right) & \text{if } j < m+1 \\ 0 & \text{if } j > m. \end{cases} \tag{B.6}$$

Here we have taken advantage of the fact that $\mathsf{B}_{2l,j}(0, x_2, x_3, \dots, x_{m-j+1}) = 0$ for all $j \geq l+1 > 0$. This can be seen by setting $j \geq l+1$, $k_1 = 0$, and taking non-negative integers $0 = k_1, \dots, k_{2l-j+1}$ which satisfy (B.2) with $m = 2l$. Then clearly $\sum_{i=2}^{2l-j+1} k_i = j$, but the second condition in (B.2) gives

$$\sum_{i=2}^{2l-j+1} i k_i = \sum_{i=2}^{2l-j+1} (i-2)k_i + 2\sum_{i=2}^{2l-j+1} k_i \geq 2(l+1) > 2l, \tag{B.7}$$

which is a contradiction. So we must have $k_1 \neq 0$, and hence the terms of $\mathsf{B}_{2l,j}$ depend on k_1.

Appendix C. Asymptotics of Jacobi theta functions $\vartheta_1, \vartheta_2, \vartheta_3$ and their derivatives

Here we present asymptotic data for the theta functions and their derivatives introduced and explored in Section 3 for $s > 0$ and as $s \searrow 0$. For further details *see* [11, 24] and [4]. Indeed starting from ϑ_1 we have

$$\vartheta_1(s) = \sqrt{\pi/s} + O(e^{-1/s}), \tag{C.1}$$

$$\vartheta_1^{(m)}(s) = (-1)^m \Gamma(m+1/2) s^{-m-1/2} + O(e^{-1/s}). \tag{C.2}$$

For ϑ_2 and ϑ_3, we use the well-known Bernoulli numbers B_{2j} [4, 11, 24] to write

$$\vartheta_2(s) \sim \frac{1}{s} + \sum_{j=0}^{\infty} \frac{s^j}{j!} \frac{(-1)^j}{j+1} \left(1 - 2^{-2j-1}\right) B_{2j+2}, \tag{C.3}$$

$$\vartheta_2^{(m)}(s) \sim \frac{(-1)^m m!}{s^{m+1}} + \sum_{j=m}^{\infty} \frac{s^{j-m}}{(j-m)!} \frac{(-1)^j}{j+1} \left(1 - 2^{-2j-1}\right) B_{2j+2} \tag{C.4}$$

and in a similar way

$$\vartheta_3(s) \sim \frac{1}{s} + \sum_{j=0}^{\infty} \frac{s^j}{j!} \frac{(-1)^j}{j+1} B_{2j+2}, \tag{C.5}$$

$$\vartheta_3^{(m)}(s) \sim \frac{(-1)^m m!}{s^{m+1}} + \sum_{j=m}^{\infty} \frac{s^{j-m}}{(j-m)!} \frac{(-1)^j}{j+1} B_{2j+2}. \tag{C.6}$$

References

1. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series **55**, 1983.
2. G.E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications **2**, Cambridge University Press, 1976.
3. G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, 1999.
4. R.O. Awonusika, A. Taheri, *On Jacobi polynomials* $(P_k^{(\alpha,\beta)} : \alpha, \beta > -1)$ *and Maclaurin spectral functions on rank one symmetric spaces*, J. Anal. **25** (2017) 139-166.
5. R.O. Awonusika, A. Taheri, *On Gegenbauer polynomials and coefficients* $c_j^l(\nu)$, Results in Math. **72** (2017) 1359-1367.
6. R.O. Awonusika, A. Taheri, *A spectral identity on Jacobi polynomials and its analytic implications*, Canad. Math. Bull., To appear 2018.
7. D. Bakry, I. Gentil, M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Grundlehren der Mathematischen Wissenschaften **348**, Springer, 2008.
8. W. Beckner, *Sobolev inequalities, the Poisson semigroup and analysis on the sphere* $\mathbb{S}^n$, Proc. Nat. Acad. Sci. USA **89** (1992) 4816-4819.
9. E.T. Bell, *Exponential polynomials*, Ann. of Math. **35** (1934) 258-277.
10. M. Berger, P. Gauduchon, E. Mazet, *Le spectre dúne variétè Riemannienne*, Springer, 1971.
11. R.S. Cahn, J.A. Wolf, *Zeta functions and their asymptotic expansions for compact symmetric spaces of rank one*, Comm. Math. Helv. **51** (1976) 1-21.
12. S. Day, A. Taheri, *A formulation of the Jacobi coefficients via Bell polynomials*, Adv. Op. Th. **2** (2017) 506-515.
13. S. Day, A. Taheri, *Semigroup asymptotics, Funk-Hecke identity and the Gegenbauer coefficients associated with the Spherical Laplacian*, Rocky Mount. J. Math., To appear 2018.
14. J. Dolbeault, M.J. Esteban, M. Kowalczyk, M. Loss, *Sharp interpolation inequalities on the sphere: New methods and consequences*, Chinese Ann. Math. **34** (2013) 99-112.
15. N. Dunford, J.T. Schwartz, *Linear Operators* **I-III**, Wiley Classics in Mathematics, Wiley-Blackwell, New Ed 1988.
16. A. Erdélyi, *Higher Transcendental Functions*, McGraw-Hill, 1953.
17. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications **96**, Cambridge University Press, 2004.
18. S. Helgason, *Eigenspaces of the Laplacian; integral representations and irreducibility*, J. Funct. Anal. **17** (1974) 328-353.
19. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, 1978.
20. T.H. Koornwinder, *The addition formula for Jacobi polynomials:* **I** *Summary of results*, Indag. Math. **34** (1974) 188-191.
21. T.H. Koornwinder, *A new proof of a Paley-Wiener type theorem for the Jacobi transform*, Ark. Matematik **13** (1975) 145-159.
22. J. Letessier,G. Valent, J. Wimp, *Some Differential Equations Satisfied by Hypergeometric Functions*, International Series of Numerical Mathematics **119**, Birkhauser, 1994.

23. H. McKean, I.M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geom. **1** (1967) 43-69.
24. H. Mulholland, *An asymptotic expansion for* $\sum(2n+1)e^{-\sigma(n+1/2)^2}$, Proc. Camb. Phil. Soc. **24** (1928) 280-289.
25. B. Osgood, R. Phillips, P. Sarnak, *Extremals and determinants of Laplacians*, J. Funct. Anal. **80** (1988) 148-211.
26. I. Polterovich, *Heat invariants of Riemannian manifolds*, Israel J. Math. **119** (2000) 239-252.
27. P. Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. **110** (1987) 113-120.
28. R. Seeley, *Complex powers of an elliptic operator*, In: Singular Integrals, Proc. Sympos. Pure Math., Chicago, **III**, pp. 288-307, Amer. Math. Soc. Providence, R.I., 1966.
29. A. Taheri, *Function Spaces and Partial Differential Equations* **I** & **II**, Oxford Lecture Series in Mathematics and its Applications **40** & **41**, Oxford University Press, 2015.
30. N.J. Vilenkin, *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs **22**, Amer. Math. Soc., 1968.
31. V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group*, Springer Monographs in Mathematics, 2009.
32. G. Warner, *Harmonic Analysis on Semisimple Lie Groups* **I** & **II**, Springer, 1972.

[1]Department of Mathematics, University of Sussex, Brighton, East Sussex.
E-mail address: s.bond@sussex.ac.uk

[2]Department of Mathematics, University of Sussex, Brighton, East Sussex.
E-mail address: a.taheri@sussex.ac.uk